Applied Differential Equations 2250-1
Sample Midterm Exam
Fall 2001
Exam Date: Friday, Nov 9, 2001

1. (Separable Equations) Solve the separable problem below for the *implicit* and *explicit* solutions.

$$[\cos(y)]\, y' = \frac{3x^2}{\sqrt{1+x^3}} \cot y + 4 \sin x \frac{\cos y}{\sin y}.$$

Answer:

$-\cos y(x) = 2(1+x^3)^{1/2} - 4\cos x + C.$

$y(x) = \cos^{-1}\left(4\cos x - 2\sqrt{1+x^3} - C\right).$

2. (Variation of Parameters) Determine the general solution $y = y_c + y_p$ for the equation $6y'' - 5y' - 4y = \pi e^{-x^2}$ by the method of variation of parameters. Leave the answer for y_p in unevaluated integral form.

Answer:

Let $f(t) = \pi e^{-t^2}$. Then $y_h(x) = c_1\, e^{4x/3} + c_2\, e^{-x/2}$,

$y_p(x) = \dfrac{1}{6}\displaystyle\int_0^x k(x,t) f(t) dt,$

$k(x,t) = \dfrac{6}{11}\, e^{4(x-t)/3} - \dfrac{6}{11}\, e^{-(x-t)/2}.$

3. (Homogeneous systems) Solve the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Answer:

The reduced row echelon form of the coefficient matrix has one zero row, so there is one free variable and two lead variables. The solution is $x = -t$, $y = t$, $z = t$ for $-\infty < t < \infty$. The reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. (Damped Oscillator) Solve by undetermined coefficients the damped oscillator problem

$$x'' + 4x' + 5x = \sin(t), \quad x(0) = 0, \quad x'(0) = 0.$$

Report the **transient response** and also the **steady state response** of the mechanical system. Classify the unforced equation as *overdamped*, *underdamped* or *critically damped*.

Answer:

$$x(t) = -\frac{1}{8}\cos(t) + \frac{1}{8}\sin(t) + \frac{1}{8}e^{-2t}\cos(t) + \frac{1}{8}e^{-2t}\sin(t).$$

The transient solution is $\frac{1}{8}e^{-2t}\cos(t) + \frac{1}{8}e^{-2t}\sin(t)$.

The steady-state solution is $-\frac{1}{8}\cos(t) + \frac{1}{8}\sin(t)$. The unforced equation in underdamped.

Undetermined coefficients. The homogeneous solution $x_h(t)$ and the trial solution $X(t)$ for particular solution $x_p(t)$ are

$$x_h(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t),$$

$$X(t) = a\cos(t) + b\sin(t) \quad \text{(trial solution)}$$

The constants a and b are found by the method of undetermined coefficents to be $a = -\frac{1}{8}$, $b = \frac{1}{8}$. The constants c_1 and c_2 are determined from the initial conditions $x(0) = 0$, $x'(0) = 0$ to be $c_1 = c_2 = \frac{1}{8}$.

The equilibrium method uses $y = \Im(Ye^{it})$ and $[(D+i)^2 + 4(D+i) + 5]Y = 1$. Then $Y = 1/(4+4i) = (1-i)/8$ and $y = \Im(Ye^{it}) = -\frac{1}{8}\cos(t) + \frac{1}{8}\sin(t)$.