Applied Differential Equations 2250-1
Midterm Exam
Fall 2001
Exam Date: Friday, November 9, 2001

Instructions. Calculators, hand-written or computer-generated notes are allowed, including xerox copies of tables or classroom xerox notes. Books are not allowed.

1. **(Separable Equations)** Solve the separable problem below for the *implicit* and *explicit* solutions. Distinguish equilibrium and non-equilibrium solutions as needed.

$$y' = \frac{x^3}{\sqrt[3]{1+3x^4}}\cos^2 2y + \csc x \cot x \cos^2 2y.$$

Answer:

The equilibrium solutions are where $\cos 2y = 0$, that is, $2y = (2n+1)\pi/2$ with $n = 0, \pm1, \pm2, \ldots$. The non-equilibrium solutions satisfy $y' \sec^2 2y = \frac{x^3}{\sqrt[3]{1+3x^4}} + \csc x \cot x$. This equation is the separated form. Quadrature gives the *implicit solution* $\frac{1}{2}\tan 2y(x) = \frac{1}{8}(1+3x^4)^{2/3} - \csc x + C$.

The *explicit solution* is $y(x) = \frac{1}{2}\arctan\left(\frac{1}{4}(1+3x^4)^{2/3} - 2\csc x + C\right)$.

2. **(Variation of Parameters)** Determine the general solution $y = y_h + y_p$ for the equation $16y'' + 10y' - 21y = \pi^2 \sin(\ln(1+x))$ by the method of variation of parameters. Leave the answer for y_p in unevaluated integral form.

Answer:

The solution is $y = y_h + y_p$. Let $f(t) = \pi^2\sin(\ln(1+x))$. Then $16r^2 + 10r - 21 = 0$ has roots $r = 7/8$, $r = -3/2$ and case 1 of the recipe gives $y_h(x) = c_1 e^{7x/8} + c_2 e^{-3x/2}$. Let $y_1 = e^{7x/8}$, $y_2 = e^{-3x/2}$, $\Delta_1 = y_1(t)y_2(x) - y_1(x)y_2(t)$, $\Delta = y_1(t)y_2'(t) - y_1'(t)y_2(t)$, $k(x,t) = \Delta_1/\Delta$. Then
$k(x,t) = -(8/19)e^{-3(x-t)/2} + (8/19)e^{7(x-t)/8}$ and $y_p(x) = \frac{1}{16}\int_0^x k(x,t)f(t)dt$.

3. **(Homogeneous systems)** Assume a, b are real numbers and $b \neq 0$. Solve the system below. Report the parametric solution and identify the lead and free variables.

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ b & 2 & 0 & -1 \\ b & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix}.$$

Answer:

The reduced echelon form of the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/b & -a/b \\ 0 & 1 & 0 & -1 & a \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix has one zero row, so there are two free variables z, w and two lead variables x, y. The parametric solution is $x = -t/b - a/b$, $y = t + a$, $z = s$, $w = t$ for $-\infty < s, t < \infty$.

4. (Oscillations) Solve by undetermined coefficients the oscillation problem

$$9x'' + 4x' + x = 5\cos(t), \quad x(0) = 0, \quad x'(0) = 0.$$

Report the **transient response** and also the **steady state response** of the mechanical system. Classify the unforced equation as *overdamped*, *underdamped* or *critically damped*.

Answer:

$$x(t) = \frac{1}{2}e^{-2t/9}\cos(\sqrt{5}t/9) - \frac{\sqrt{5}}{4}e^{-2t/9}\sin(\sqrt{5}t/9) - \frac{1}{2}\cos(t) + \frac{1}{4}\sin(t).$$

The **transient solution** is $\frac{1}{2}e^{-2t/9}\cos(\sqrt{5}t/9) - \frac{\sqrt{5}}{4}e^{-2t/9}\sin(\sqrt{5}t/9)$.

The **steady-state solution** is $-\frac{1}{2}\cos(t) + \frac{1}{4}\sin(t)$. The unforced equation is **underdamped**.

Classical undetermined coefficients. Let

$$X(t) = a\cos(t) + b\sin(t) \quad \text{(trial solution)}$$

The constants a and b are found by the method of undetermined coefficents to be $a = -\frac{1}{2}$, $b = \frac{1}{4}$.

Kümmer's method. Let $x = \mathrm{Re}(Xe^{it})$ and $[9(D+i)^2+4(D+i)+1]X = 5$. The *equilibrium method* implies $X = 5/(-8+4i) = 5(-8-4i)/80 = -1/2 - i/4$ and $x(t) = \mathrm{Re}(Xe^{it}) = -\frac{1}{2}\cos(t) + \frac{1}{4}\sin(t)$.

Homogeneous solution. Recipe case 3 uses $9r^2 + 4r + 1 = 0$ and then

$$x_h(t) = c_1e^{-2t/9}\sin(\sqrt{5}t/9) + c_2e^{-2t/9}\sin(\sqrt{5}t/9).$$

Initial value problem. The constants c_1 and c_2 are determined from the initial conditions $x(0) = 0$, $x'(0) = 0$ to be $c_1 = \frac{1}{2}$, $c_2 = -\frac{\sqrt{5}}{4}$.