Math 2250
Maple Project 3A
April 2002

The third project is divided into two parts. Part A is on integration theory for improper integrals and some related calculations that are used in Laplace transform theory. The basic work for Part A is described in Edwards-Penney, Chapter 10. Part B consists of additional examples, in particular therein are substantial applications of the Laplace theory applied to differential equations with step function inputs and impulsive inputs. Equal credit is given for the two parts. Please submit Part A and Part B in 10 days.

3A.1. _____ Evaluation of improper integrals
3A.2. _____ Evaluation of improper integrals
3A.3. _____ Inverse Laplace
3A.4. _____ Differential equations
3A.5. _____ Differential equations
3B.1. _____ Waves
3B.2. _____ Sawtooth wave forced oscillator
3B.3. _____ Triangular wave forced oscillator
3B.4. _____ Half-wave sine rectifier forced oscillator
3B.5. _____ Resonance in a forced oscillator

Laplace Transform.

The *Direct Laplace transform* of $f(t)$ is the integral

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

There are `maple` methods for evaluating the integral symbolically. See the examples below for details.

Problem 3A.1. (Evaluation of improper integrals)

Derive the following formulas and then check the answers using `maple`. In each case, $s > s_0$ for sufficiently large s_0.

$$\int_0^\infty e^{-st} f(t)dt = \begin{cases} \dfrac{1}{s} & f(t) = 1, \\ \dfrac{1}{s-a} & f(t) = e^{at}, \\ \dfrac{1}{s^2} & f(t) = t, \\ \dfrac{2}{s^3} & f(t) = t^2. \end{cases}$$

Problem 3A.2. (Evaluation of improper integrals)

Evaluate the improper integrals below using `maple`, checking the given answer.

$\mathcal{L}\left(e^{-2t}\right) = \dfrac{1}{s+2}$,

$\mathcal{L}\left(e^{3t}\right) = \dfrac{1}{s-3}$,

$\mathcal{L}\left(6\cos(3t)\right) = \dfrac{6s}{s^2+9}$,

$\mathcal{L}\left(e^{-t}\cos(4t)\right) = \dfrac{s+1}{(s+1)^2+16}$,

$\mathcal{L}\left(te^{t}\cos(t)\right) = \dfrac{s(s-2)}{(s+1)^2+1}$,

$\mathcal{L}\left(\sinh(t)\right) = \dfrac{1}{s^2-1}$,

$\mathcal{L}\left(e^{-2t}\sin(3\pi t)\right) = \dfrac{3\pi}{(s+2)^2+9\pi^2}$.

Inverse Laplace Transform.

The *Inverse Laplace transform* of $F(s)$ is the solution $f(t)$ of the equation

$$\int_0^\infty e^{-st} f(t)dt = F(s).$$

This is basically a table lookup, provided the integration table is rich enough to contain the answer for $f(t)$. The example in the `maple` notes below shows how to do it.

Problem 3A.3. (Inverse Laplace)

Find $f(t)$ in the equations below using `maple`.

$\mathcal{L}\left(f(t)\right) = \dfrac{1}{4s^2}$,

$\mathcal{L}\left(f(t)\right) = \dfrac{s-1}{s^2}$,

$\mathcal{L}\left(f(t)\right) = \dfrac{2s^3-s^2}{(4s^2-4s+5)^2}$,

$\mathcal{L}\left(f(t)\right) = \dfrac{2s}{s^2+4}(e^{-\pi s} - e^{-2\pi s})$,

$\mathcal{L}\left(f(t)\right) = \dfrac{e^{-1/s}}{\sqrt{s}}$.

Differential equations.

The solution of a differential equation in `maple` is a powerful aid to computation. Most intuition comes from hand-built solution methods, but to get the right answer, an assist is sometimes essential. The basic syntax is communicated by the example in the notes below.

Problem 3A.4. (Differential equations)

Solve with `maple` methods the following differential equation and show the answer is $x = 2te^{-t/5}\sin(3t)$.

$$x'' + 0.4x' + 9.04x = 12e^{-t/5}\cos(3t),$$
$$x(0) = 0, \quad x'(0) = 0.$$

Problem 3A.5. (Differential equations)

Solve with `maple` methods the following differential equation and show the answer is $x = t\sin(3t)$.

$$x'' + 9x = 6\cos(3t),$$
$$x(0) = 0, \quad x'(0) = 0.$$

Notes $3A.1$. The *improper integral* is defined by the limit relation

$$\int_0^\infty F(t)dt = \lim_{N\to\infty}\int_0^N F(t)dt.$$

The integrand $F(t)$ in all cases are continuous and therefore the finite integral always makes sense.

The first two integrals are done by direct evaluation while the third and fourth require integration by parts. For example,

$$\int_0^N e^{-st}[t]dt = uv|_0^N - \int_0^N vdu$$

where

$$u = t, \quad dv = e^{-st}dt.$$

Evaluating gives

$$\begin{aligned}\int_0^N e^{-st}[t]dt &= t\frac{e^{-st}}{-s}\Big|_0^N - \int_0^N \frac{e^{-st}}{-s}dt\\ &= N\frac{e^{-sN}}{-s} + \int_0^N \frac{e^{-st}}{s}dt\\ &= N\frac{e^{-sN}}{-s} - \frac{e^{-sN}}{s^2} + \frac{1}{s^2}.\end{aligned}$$

In the limit as $N \to \infty$, the exponential terms are carried to zero and the answer is

$$\int_0^\infty e^{-st}[t]dt = \frac{1}{s^2}.$$

To check the answer using `maple`, apply the following example.

```
F:=t->t^2*exp(-s*t):
int(F(t),t=0..infinity);
```

In `maple` V5, the response is

```
Definite integration: Can't determine
if the integral is convergent.
Need to know the sign of --> s
Will now try indefinite integration
and then take limits.
```

$$\lim_{t\to\infty}\frac{2}{s^3} + \frac{-t^2e^{-st}}{s} - \frac{2te^{-st}}{s^2}.$$

From this display, it is usually easy to decide upon the answer, by setting to zero the exponential terms with negative exponent.

To automate `maple` and compute the limit, add assumptions about the variables to `maple`'s internals, as follows:

```
F:=t->t^2*exp(-s*t):
assume(s>0):
int(F(t),t=0..infinity);
```

`maple` Notes $3A.2$.

The example below shows how to do it in `maple` V5.

```
with(inttrans):
L:=f->eval(laplace(f,t,s)):
L(t*exp(t)*cos(t));
```

`maple` Notes $3A.3$.

The example below shows how to do it in `maple` V5.

```
with(inttrans):
LI:=F->eval(invlaplace(F,s,t)):
LI((1/4)/s^2);
```

The answer is $f(t) = (1/4)t$. The answer means the Laplace integral of $(1/4)t$ equals $(1/4)/s^2$.

`maple` Notes $3A.4$ and $3A.5$.

```
f:=t->6*exp(-t/5)*cos(3*t):
de:=diff(x(t),t,t) +
    0.4*diff(x(t),t)+9.04*x(t) = f(t):
ic:=x(0)=0,D(x)(0)=0:
dsolve({de,ic},x(t),method=laplace);
```

The displayed answer is

$$x(t) = te^{-t/5}\sin(3t).$$

Math 2250
Maple Project 3B
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Part B consists of applications to differential equations with periodic step function inputs. Please submit Part B with Part A.

Heaviside's function. Let $H(t)$ denote Heaviside's unit step function,

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

This *unit step function* is equivalent to a *switch* turned on at $t = 0$. A switch turned on at $t = a$ is given by $H(t-a)$. Engineering functions with switches can be constructed from H by taking combinations. For example, a switch turned on for interval $a \leq t < b$ is given by the combination

$$H(t-a) - H(t-b) = \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise.} \end{cases}$$

Complex switches occur naturally in applications, for example, a voltage of 12 volts turned on in a circuit at time $t = 0.5$ is given by $V = 12H(t-0.5)$.

Periodic switching and Heaviside's function. Periodic switching functions can be written in terms of H by series methods. For base function F defined on $0 \leq t \leq T$, a T-periodic extension f to $0 \leq t < \infty$ is defined by the relations

$$\begin{aligned} f(t) &= F(t-nT) \quad \text{for} \quad nT \leq t \leq nT+T \\ &= F(t)[H(t) - H(t-T)] \\ &\quad + F(t-T)[H(t-T) - H(t-2T)] + \cdots \\ &= \sum_{k=0}^{\infty} F(t-kT)[H(t-kT) - H(t-kT-T)]. \end{aligned}$$

On computers, F should be defined to be zero outside $0 \leq t \leq T$.

The series can be written with just positive indices:

$$\begin{aligned} f(t) &= F(t)[1 - H(t-T)] \\ &\quad + \sum_{k=1}^{\infty} F(t-kT)[H(t-kT) - H(t-kT-T)]. \end{aligned}$$

Problem 3B.1. (Waves)

Let $T = 2\pi$ be the period for the following finite T-periodic waves. Code the waves in `maple` and then plot them for 10 periods $0 \leq t \leq 10T$.

Square Wave.

$$f(t) = \sum_{k=0}^{10} (H(t-kT) - H(t-kT-T/2)).$$

Sawtooth Wave.

$$f(t) = \sum_{k=0}^{10} \left| \frac{t}{T} - k \right| (H(t-kT) - H(t-kT-T)).$$

Triangular Wave.

$$f(t) = \sum_{k=0}^{10} \left(\frac{2t}{T} - 2k \right) (H(t-kT+T/2) - H(t-kT-T/2))$$

Half-Wave Sine Rectifier.

$$f(t) = \sum_{k=0}^{10} \sin(t)(H(t-kT) - H(t-kT-T/2))$$

Full-Wave Sine Rectifier.

$$f(t) = \sum_{k=0}^{10} |\sin(t)|(H(t-kT) - H(t-kT-T))$$

Problem 3B.2. (Sawtooth wave forced oscillator)

Plot the solution of the problem

$$x''(t) + x(t) = f(t), \quad x(0) = x'(0) = 0,$$

where $f(t)$ is the sawtooth wave of $3B.1$, over $t = 0$ to $t = 10T$, $T = 2\pi$.

Problem 3B.3. (Triangular wave forced oscillator)

Plot the solution of the problem

$$x''(t) + x(t) = f(t), \quad x(0) = x'(0) = 0,$$

where $f(t)$ is the triangular wave of $3B.1$, over $t = 0$ to $t = 10T$, $T = 2\pi$.

Problem 3B.4. (Half-wave sine rectifier forced oscillator)

Plot the solution of the problem

$$x''(t) + x(t) = f(t), \quad x(0) = x'(0) = 0,$$

where $f(t)$ is the half-wave sine rectifier of $3B.1$, over $t = 0$ to $t = 10T$, $T = 2\pi$.

Problem 3B.5. (Resonance in a forced oscillator)

The solution of the problem

$$x''(t) + a^2x(t) = f(t), \quad x(0) = x'(0) = 0$$

may become unbounded at $t \to \infty$ because the natural frequency of the *input* $f(t)$ exactly matches the natural frequency a of the unforced oscillator $x'' + a^2x = 0$. Let $a = 1$ and $T = 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$. Determine by graphing the solution $x(t)$ which combinations of $f(t)$ and T produce unbounded solutions (resonance), where f is one of the five types of waves given in $3B.1$. Present a table of answers, wave against resonant period T.

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maple Notes $3B.1$. To code it using maple, modify the following example, which is for the *square wave*. In maple 5, the Heaviside function is not defined at $t = 0$, and this causes problems with graphics, so a local version of the Heaviside function must be defined, as below. In other versions of maple, this might not be required.

```
t:='t':f:='f':T:=2*Pi:
H:=t-> piecewise(t<0,0,1):
squarewave:=
sum(H(x-k*T)-H(x-k*T-T/2),k=0..10):
f:=t->evalf(subs(x=t,squarewave)):
opts:=style=point:
plot(f,0..10*T,opts);
```

The *sawtooth wave* and *triangular wave* are coded similarly. The changes occur in only one line above:

```
sawtoothwave:=sum((x/T-k)*
(H(x-k*T)-H(x-k*T-T)),k=0..10):
trianglewave:=sum((2*x/T-2*k)*
(H(x-k*T+T/2)-H(x-k*T-T/2)),k=0..10):
```

The figures for the *half-wave sine rectifier* and the *full-wave sine rectifier* look considerably more realistic with the plot options below.

```
opts:=style=line,scaling=constrained:
plot(f,0..10*T,opts);
```

The half-wave sine rectifier is a multiple $\sin(t)$ of the square wave. It may help to use the plot options above with the other waves, in order to obtain realistic plots.

---

maple Notes $3B.2$. To code it using maple, modify the following example, which is for the *square wave*. The code changes slightly from the plot code of $3B.1$, starting the sum index at $k = 1$ instead of $k = 0$. The term for $k = 0$, namely $H(x) - H(x - T/2)$, is replaced by $1 - H(x - T/2)$, to prevent internal errors in the package odeplot. Option numpoints=90 may have to be changed, depending on the maple version (try removing it completely).

```
with(plots):
t:='t':x:='x':pp:='pp':T:=2*Pi:
H:=t-> piecewise(t<0,0,1):
squarewave:=1-H(x-T/2)+
sum(H(x-k*T)-H(x-k*T-T/2),k=1..10):
f:=t->evalf(subs(x=t,squarewave)):
de:=diff(x(t),t,t)+x(t)=f(t):
ic:=x(0)=0,D(x)(0)=0:
pp:=dsolve({de,ic},x(t),type=numeric):
opts:=numpoints=90,color=black:
odeplot(pp,[t,x(t)],0..10*T,opts);
```

---

maple Notes $3B.3$. The example follows $3B.2$. Change the square wave code as follows.

```
triangularwave:=(2*x/T)*(1-H(x-T/2))+
sum((2*x/T-2*k)*
(H(x-k*T+T/2)-H(x-k*T-T/2)),k=1..10):
```

---

maple Notes $3B.4$. The example follows $3B.2$. Change the square wave code as follows.

```
halfsinewave:=sin(x)*(1-H(x-T/2))+
sum(sin(x)*
(H(x-k*T)-H(x-k*T-T/2)),k=1..10):
```

---

maple Notes $3B.5$. The work is substantial, because the plot options have to be adjusted to fit each problem. There is no way to tell if 10 or 20 periods should be used in the graphic, because unboundedness is a property that depends upon scale, and about scale there is no intutition, for these problems. Test the results with this answer: for the triangular wave, there are two periods which are non-resonant, and the others show resonance. It is left for you to report the resonance or nonresonance for the other four waves.