Math 2250
Maple Project 1
January 2001

Reference: Edwards-Penney, pages 55–57.

Due date: Wednesday next week.

The Problem. The project is to determine an approximation for the indoor temperature $u(t)$ in an unheated building. The model uses Newton cooling, insulation data and a formula for the ambient outside temperature $A(t)$.

Assumptions and Notation. Let the daily temperature in Salt Lake City vary from a low of $m = 25$ degrees to a high of $M = 55$ degrees with the low and high at 3am and 3pm, respectively. The building is assumed insulated, the average insulation factor being represented by a Newton cooling constant k, usually $0.2 \le k \le 0.5$. The ambient outside temperature is called $A(t)$ and the inside temperature is called $u(t)$, where t is in hours measured from midnight $t = 0$. The number u_0 is the temperature at midnight. The model assumes no heat heat sources or sinks are present inside the building, e.g., the heat pump is off.

Problem 1.1. (Newton Cooling Model)

Let $A(t)$ be the ambient outside temperature. Let k denote a constant. Justify from Newton's law of cooling that a model for the indoor temperature $u(t)$ is given by

$$u'(t) + ku(t) = kA(t), \quad u(0) = u_0. \tag{1}$$

Problem 1.2. (Ambient Temperature Model)

Let $M = 55$, $m = 25$ and $\omega = \pi/12$. Verify that the ambient temperature formula

$$A(t) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m)\cos\omega(t - 3)$$

satisfies $M = \max A(t) = A(15)$, $m = \min A(t) = A(3)$ and $A(t)$ is periodic of period 24 hours. Plot $A(t)$ on $0 \le t \le 24$ and label the maxima and minima.

Problem 1.3. (Steady-state Periodic Solution)

Derive a formula for the steady-state periodic solution u_{SS} of equation (1). Check the answer in `maple`.

Problem 1.4. (Indoor Temperature $u(t)$)

Give a formula for $u(t)$ in terms of the homogeneous solution u_h and the steady-state solution u_{SS}. The answer depends on u_0 and k. Make on one graphic four plots of $u(t)$ for $k = 0.3$ and $u_0 = 40$, $u_0 = 50$, $u_0 = 60$, $u_0 = 70$ for $0 \le t \le 72$ hours. Label the maxima and minima.

Problem 1.5. (Indoor-Outdoor Variation)

Compare in a graphic the indoor and outdoor temperature oscillations over a 48-hour period assuming $k = 0.3$, $u_0 = 67$. Display a table of low and high indoor temperatures, the indoor temperature variation and the phase delay.

Problem 1.6. (Freezing Pipes)

Assume the insulation constant k ranges from 0.2 to 0.5. Suppose the inside temperature is 67 degrees at midnight when the furnace is turned off. In what range of hours will the indoor temperature reach 31 degrees? Explain your answers mathematically and illustrate with a graphic.

References and problem notes.

An on-line reference is `2250template1.pdf` at URL `http://www.math.utah.edu/~korevaar/`.

Notes on 1.1: A careful mathematical derivation is expected, using words and Newton's cooling law, which reads

> *The rate of change of the indoor temperature is proportional to the difference between the ambient and indoor temperatures, that is,* $u' = k(u - A)$.

Notes on 1.2: The maximum and minimum occur when the cosine term is smallest and largest, which is when the argument of the cosine is π or 0, respectively. To show that $A(t)$ is periodic of period 24, observe that sums of T-periodic functions are T-periodic, and that $\cos\omega t$ is known from trigonometry to have period $2\pi/\omega$.

Notes on 1.3: The integrating factor method for linear equations applies to find the general solution by these steps:

$u' + ku = kA(t)$	Copy equation (1). The integrating factor is e^{kt}.
$\dfrac{(e^{kt}u)'}{e^{kt}} = kA(t)$	Replace the left side.

$$(e^{kt}u)' = kA(t)e^{kt} \qquad \text{Clear fractions.}$$

The general solution u is then found by integrating both sides and dividing to isolate u:

$$e^{kt}u = u_0 + \int_0^t kA(r)e^{kr}\,dr$$
$$u = u_0 e^{-kt} + e^{-kt}\int_0^t kA(r)e^{kr}\,dr$$

The steady-state solution is found after integration by dropping all terms that contain e^{-kt}.

$$u_{SS} = 40 - \frac{15k}{k^2+\omega^2}(k\cos\omega(t-3) + \omega\sin\omega(t-3)).$$

The code used in `maple` to find the integral is

```
M:=55:m:=25:omega:='omega':
A:=t->(M+m)/2-(M-m)*(1/2)*cos(omega*(t-3));
u:=t->u0*exp(-k*t)+
      int(exp(k*r-k*t)*k*A(r),r=0..t);
simplify(u(t));
```

To *check the answer*, use `maple` as follows. The result should be *zero* or a trigonometric expression that reduces to zero.

```
# Test LHS=RHS for y'+ky=kA.
M:=55:m:=25:omega:='omega':
A:=t->(M+m)/2-(M-m)*(1/2)*cos(omega*(t-3));
y:=t->40-15*k*(k*cos(omega*(t-3))
     +omega*sin(omega*(t-3)))/(k^2+omega^2):
LHS:=diff(y(t),t)+k*y(t):
RHS:=k*A(t);
simplify(LHS-RHS);
```

Notes on 1.4: The general solution is $u = u_h + u_p$, by the principle of superposition. The notation: u_h is the solution of the homogeneous equation $u' + ku = 0$ and u_p is a particular solution of the nonhomogeneous equation $u' + ku = kA$. By Problem 1.3, u_p can be taken to be u_{SS}. The work left is to solve $u' + ku = 0$ and then add that answer to u_{SS}. The result is an answer that contains one arbitrary constant C. Finally, use $u(0) = u_0$ to write C in terms of k and u_0.

For the plot, set $k = 0.3$ and $\omega = \pi/12$. For each of the four values of u_0 there is a formula for u; denote these formulas by u_1, u_2, u_3, u_4. The `maple` code below shows how to put the four plots onto one graphic.

```
# Assume u1, u2, u3, u4 already defined.
plot({u1(t),u2(t),u3(t),u4(t)},t=0..72);
```

Notes on 1.5: The work in earlier problems supplies the formulas for $u_{SS}(t)$ and $A(t)$, using $k = 0.3$, $u_0 = 67$, $\omega = \pi/12$. The plot can be done in `maple` as follows.

```
# Assume uss(t) and A(t) already defined.
plot({uss(t),A(t)},t=0..48);
```

The table is produced by hand, although it may be just as easy to make it in `maple`. The maximum and minimum can be found using maple functions `maximize` and `minimize`. The indoor temperature variation is just the maximum minus the minimum. The phase shift is computed as $|T_2 - T_1|$, where $A(T_1) = \max A(t)$ and $u_{SS}(T_2) = \max u_{SS}(t)$. Look at the graphic to find sane answers for T_1 and T_2. Read the textbook for a more complete discussion of the ideas.

Notes on 1.6: The solution $u(t)$ when $u_0 = 67$ is really a function of t and k. If it was possible to graph u with the arbitrary constant left in the formula, then this problem would be easy. But that is not a feature of `maple`. Instead, you have to choose a lot of different values for k in the range $0.2 \le k \le 0.5$, and examine all the graphs, in order to gain intuition about the answer.

A clever visualization idea is to define $f(t,k)$ by the formula for u, then do a 3D-graphic of f on the domain $0 \le t \le 72$, $0.2 \le k \le 0.5$. We are trying to find out the times t which correspond to $f = 31$. But $f = 31$ is a plane parallel to the tk-plane in this 3D-plot, so the graphic is useful for checking answers.

```
omega:=Pi/12:k:='k':
uss:=t->40-15*k*(k*cos(omega*(t-3))
     +omega*sin(omega*(t-3)))/(k^2+omega^2):
C:=67-uss(0):
f:=unapply(C*exp(-k*t)+uss(t),(t,k));
plot3d(f(t,k),t=0..72,k=0.2..0.5);
```

A better choice of computer algebra assist in this problem can be found in `maple`'s function `implicitplot`. This function can plot the equation $f(t,k) = 31$ over the domain $0 \le t \le 72$, $0.2 \le k \le 0.5$. From this plot, the question is easily answered.

```
# Assume f(t,k) already defined.
with(plots):
implicitplot(f(t,k)=31,t=0..72,k=0.2..0.5);
```

A repeated plot using a much smaller time domain gives more precise information. Physically, 31 degrees is reached in 4-10 hours after furnace shutdown.