## Chapter 8

## Linear Algebra

The subject of linear algebra includes the solution of linear equations, a topic properly belonging to college algebra. The applied viewpoint taken here is motivated by the study of mechanical systems and electrical networks, in which the notation and methods of linear algebra play an important role.

Section 8.1 An introduction to linear equations requiring only a col$a x+b y=e \quad$ lege algebra background: parametric solutions, reduced $c x+d y=f \quad$ echelon systems, basis, nullity, rank and nullspace.

Section 8.2 Matrix-vector notation is introduced, especially designed $A X=b \quad$ to prepare engineers and scientists to use computer user $Y^{\prime}=A Y \quad$ interfaces from matlab and maple. Topics: matrix equations, change of variable, matrix multiplication, row operations, reduced row echelon form, matrix differential equation.
Section 8.3 Eigenanalysis for matrix equations. Applications to dif-
$A X=\lambda X$ ferential equations. Topics: eigenanaysis, eigenvalue, $P^{-1} A P=D \quad$ eigenvector, eigenpair, ellipsoid and eigenanalysis, change of basis, diagonalization, uncoupled system of differential equations, coupled systems.

### 8.1 Linear Equations

Given numbers $a_{11}, \ldots, a_{m n}, b_{1}, \ldots, b_{m}$ and a list of unknowns $x_{1}, x_{2}$, $\ldots, x_{n}$, consider the nonhomogeneous system of $m$ linear equations in $n$ unknowns

$$
\begin{align*}
a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 n} x_{n} & =b_{1}, \\
a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 n} x_{n} & =b_{2},  \tag{1}\\
& \vdots \\
a_{m 1} x_{1}+a_{m 2} x_{2}+\cdots+a_{m n} x_{n} & =b_{m} .
\end{align*}
$$

Constants $a_{11}, \ldots, a_{m n}$ are called the coefficients of system (1). Constants $b_{1}, \ldots, b_{m}$ are collectively referenced as the right hand side, right side or RHS. The homogeneous system corresponding to system (1) is obtained by replacing the right side by zero:

$$
\begin{align*}
& a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 n} x_{n}=0, \\
& a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 n} x_{n}=0,  \tag{2}\\
& \vdots \\
& a_{m 1} x_{1}+a_{m 2} x_{2}+\cdots+a_{m n} x_{n}=0 .
\end{align*}
$$

An assignment of possible values $x_{1}, \ldots, x_{n}$ which simultaneously satisfy all equations in (1) is called a solution of system (1). Solving system (1) refers to the process of finding all possible solutions of (1). The system (1) is called consistent if it has a solution and otherwise it is called inconsistent.

In the plane ( $n=2$ ) and in 3 -space ( $n=3$ ), equations (1) have a geometric interpretation that can provide valuable intuition about possible solutions. College algebra courses often omit the discussion of no solutions or infinitely many solutions, discussing only the case of a single unique solution. In contrast, all cases are considered here.

Plane Geometry. A straight line may be represented as an equation $A x+B y=C$. Solving system (1) is the geometrical equivalent of finding all possible ( $x, y$ )-intersections of the lines represented in system (1). The distinct geometrical possibliities appear in Figures 1-3.


Figure 1. Parallel lines, no solution.

$$
\begin{aligned}
& -x+y=1, \\
& -x+y=0 .
\end{aligned}
$$



Figure 2. Identical lines, infinitely many solutions.

$$
\begin{array}{r}
-x+y=1 \\
-2 x+2 y=2
\end{array}
$$

Figure 3. Non-parallel distinct lines, one solution at the unique intersection point $P$.

$$
\begin{aligned}
-x+y & =2 \\
x+y & =0
\end{aligned}
$$

Space Geometry. A plane in $x y z$-space is given by an equation $A x+B y+C z=D$. The vector $A \vec{\imath}+B \vec{\jmath}+C \vec{k}$ is normal to the plane. An equivalent equation is $A\left(x-x_{0}\right)+B\left(y-y_{0}\right)+C\left(z-z_{0}\right)=0$, where $\left(x_{0}, y_{0}, z_{0}\right)$ is a given point in the plane. Solving system (1) is the geometric equivalent of finding all possible $(x, y, z)$-intersections of the planes represented by system (1). Illustrated in Figures 4-11 are some interesting geometrical possibilities.


Figure 4. Knife cuts an open book.
Two non-parallel planes I, II meet in a line $L$ not parallel to plane III. There is a unique point $P$ of intersection of all three planes.

$$
I: y+z=0, \quad I I: z=0, \quad I I I: x=0 .
$$

Figure 5. Triple-decker. Planes I, II, III are parallel. There is no intersection point.

$$
I: z=2, \quad I I: z=1, \quad I I I: z=0 .
$$

Figure 6. Double-decker. Planes I, II are equal and parallel to plane III. There is no intersection point.

$$
I: 2 z=2, \quad I I: z=1, \quad I I I: z=0
$$



Figure 7. Single-decker. Planes I, II, III are equal. There are infinitely many intersection points.

$$
I: z=1, \quad I I: 2 z=2, \quad I I I: 3 z=3
$$

Figure 8. Pup tent. Two non-parallel planes I, II meet in a line which never meets plane III. There are no intersection points.
$I: y+z=0, \quad I I: y-z=0, \quad I I I: z=-1$.

Figure 9. Open book. Equal planes I, II meet another plane III in a line $L$. There are infinitely many intersection points.
$I: y+z=0, \quad I I: 2 y+2 z=0, \quad I I I: z=0$.

Figure 10. Book shelf. Two planes I, II are distinct and parallel. There is no intersection point.

$$
I: z=2, \quad I I: z=1, \quad I I I: y=0
$$

Figure 11. Saw tooth. Two non-parallel planes I, II meet in a line $L$ which lies in a third plane III. There are infinitely many intersection points.
$I:-y+z=0, \quad I I: y+z=0, \quad I I I: z=0$.

Parametric Solution. The geometric evidence of possible solution sets gives rise to an algebraic problem:

What algebraic equations describe points, lines and planes?

The answer from analytic geometry appears in Table 1. In this table, $t$ and $s$ are parameters, which means they are allowed to take on any value between $-\infty$ and $+\infty$. The algebraic equations describing the geometric objects are called parametric equations.

Table 1. Parametric equations with geometrical significance.

| $x=d_{1}$, | Point. The parametric equations describe a |
| :--- | :--- |
| $y=d_{2}$, | single point. |
| $z=d_{3}$. |  |
| $x=d_{1}+a_{1} t$, | Line. The parametric equations describe a |
| $y=d_{2}+a_{2} t$, | straight line through $\left(d_{1}, d_{2}, d_{3}\right)$ with tangent |
| $z=d_{3}+a_{3} t$. | vector $a_{1} \vec{\imath}+a_{2} \vec{\jmath}+a_{3} \vec{k}$. |
| $x=d_{1}+a_{1} s+b_{1} t$, | Plane. The parametric equations describe a |
| $y=d_{2}+a_{2} s+b_{2} t$, | plane containing $\left(d_{1}, d_{2}, d_{3}\right)$. The cross product |
| $z=d_{3}+a_{3} s+b_{3} t$. | $\left(a_{1} \vec{\imath}+a_{2} \vec{\jmath}+a_{3} \vec{k}\right) \times\left(b_{1} \vec{\imath}+b_{2} \vec{\jmath}+b_{3} \vec{k}\right)$ is normal |
|  | to the plane. |

To illustrate, the parametric equations $x=2-6 t, y=-1-t, z=8 t$ describe the unique line of intersection of the three planes (details in Example 1)

$$
\begin{array}{r}
x+2 y+z=0 \\
2 x-4 y+z=8  \tag{3}\\
3 x-2 y+2 z=8
\end{array}
$$

To describe all solutions of system (1), we generalize as follows.

## Definition 1 (Parametric Equations, General Solution)

The terminology parametric equations refers to a set of equations of the form

$$
\begin{align*}
x_{1} & =d_{1}+c_{11} t_{1}+\cdots+c_{1 k} t_{k} \\
x_{2} & =d_{2}+c_{21} t_{1}+\cdots+c_{2 k} t_{k} \\
& \vdots  \tag{4}\\
x_{n} & =d_{n}+c_{n 1} t_{1}+\cdots+c_{n k} t_{k}
\end{align*}
$$

The numbers $d_{1}, \ldots, d_{n}, c_{11}, \ldots, c_{n k}$ are known constants and the variable names $t_{1}, \ldots, t_{k}$ are parameters. A general solution or parametric solution of (1) is a set of parametric equations (4) plus two additional requirements:
(5) Equations (4) satisfy (1) for $-\infty<t_{j}<\infty, 1 \leq j \leq k$.

Any solution of (1) can be obtained from (4) by specializing values of the parameters.

## Definition 2 (Minimal Parametric Solution)

Given system (1) has a parametric solution $x_{1}, \ldots, x_{n}$ satisfying (4), (5), (6), then among all such parametric solutions there is one which uses the fewest possible parameters. A parametric solution with fewest parameters is called minimal, and otherwise redundant.

## Definition 3 (Gaussian Parametric Solution)

Parametric equations (4) are called Gaussian if they satisfy

$$
\begin{equation*}
x_{i_{1}}=t_{1}, x_{i_{2}}=t_{2}, \ldots, x_{i_{k}}=t_{k} \tag{7}
\end{equation*}
$$

for distinct subscripts $i_{1}, i_{2}, \ldots, i_{k}$. The terminology is borrowed from Gaussian elimination, where such equations arise. A Gaussian parametric solution of system (1) is a set of parametric equations (4) which additionally satisfies (5), (6) and (7). See also equations (10), page 288.

For example, the plane $x+y+z=1$ has a Gaussian parametric solution $x=1-t_{1}-t_{2}, y=t_{1}, z=t_{2}$, which is also a minimal parametric solution. A redundant parametric solution of $x+y+z=1$ is $x=1-t_{1}-t_{2}-2 t_{3}$, $y=t_{1}+t_{3}, z=t_{2}+t_{3}$, using three parameters $t_{1}, t_{2}, t_{3}$.

## Theorem 1 (Gaussian Parametric Solutions)

A Gaussian parametric solution has the fewest possible parameters and it represents each solution of the linear system by a unique set of parameter values.

The theorem supplies the theoretical basis for the Gaussian algorithm to follow (page 289), because the algorithm's Gaussian parametric solution is then a minimal parametric solution. The proof of Theorem 1 is delayed until page 296. It is unlikely that this proof will be a subject of a class lecture, due to its length; it is recommended reading after understanding the examples.

Answer check algorithm. Although a given parametric solution (4) can be tested for validity manually as in Example 2 infra, it is important to devise an answer check that free of parameters. The following algorithm checks a parametric solution by testing constant trial solutions in systems (1) and (2).

Step 1. Set all parameters to zero to obtain the nonhomogeneous trial solution $x_{1}=d_{1}, x_{2}=d_{2}, \ldots, x_{n}=d_{n}$. Test it by direct substitution into the nonhomogeneous system (1).
Step 2. Consider the $k$ homogeneous trial solutions

$$
\begin{array}{cccc}
x_{1}=c_{11}, & x_{2}=c_{21}, & \ldots, & x_{n}=c_{n 1}, \\
x_{1}=c_{12}, & x_{2}=c_{22}, & \ldots, & x_{n}=c_{n 2},  \tag{8}\\
& & \\
& & \\
x_{1}=c_{1 k}, & x_{2}=c_{2 k}, & \ldots, & x_{n}=c_{n k},
\end{array}
$$

which are obtained formally by applying the partial derivatives $\partial_{t_{1}}, \partial_{t_{2}}, \ldots, \partial_{t_{k}}$ to the parametric solution (4). Verify that the trial solutions satisfy the homogeneous system (2), by direct substitution.

The trial solutions in step 2 are obtained from the parametric solution (4) by setting one parameter equal to 1 and the others zero, followed by subtracting the nonhomogeneous trial solution of step 1. The partial derivative idea computes the same set of trial solutions, and it is easier to remember.

## Theorem 2 (Answer Check)

The answer check algorithm described by steps 1-2 above tests the validity of the parametric solution (4) for all parameter values.

Proof: Although it is possible to verify the result directly (see Example 2, page 293), the reader is asked to delay the proof until Section 8.2, where matrix notation is introduced, to simplify the details of proof.

Reduced echelon systems. The plane equation $x+y+z=1$ has a Gaussian parametric solution $x=1-t_{1}-t_{2}, y=t_{1}, z=t_{2}$. We explain here how it was found, and how to generalize the idea.
The project here is to find Gaussian parametric solutions for systems (1) which have a special form called a reduced echelon system. The special form employs instead of the variable names $x_{1}, \ldots, x_{n}$ another set of $n=m+k$ variables $u_{1}, \ldots, u_{m}, v_{1}, \ldots, v_{k}$, which correspond to re-ordering or permuting the original variables:

$$
\begin{align*}
u_{1}+E_{11} v_{1}+\cdots+E_{1 k} v_{k} & =D_{1}, \\
u_{2}+E_{21} v_{1}+\cdots+E_{2 k} v_{k} & =D_{2}  \tag{9}\\
& \vdots \\
& \\
u_{m}+E_{m 1} v_{1}+\cdots+E_{m k} v_{k} & =D_{m} .
\end{align*}
$$

The numbers $D_{1}, \ldots, D_{m}, E_{11}, \ldots, E_{m k}$ are known constants. Such a system is characterized by this property:

Each of variable names $u_{1}, \ldots, u_{m}$ appears with coefficient one as the first variable in exactly one equation.

The variables $u_{1}, \ldots, u_{m}$ are called leading variables and the remaining variables $v_{1}, \ldots, v_{k}$ are called free variables. Together, these variables exhaust the original variable list $x_{1}, \ldots, x_{n}$, that is, they are a permutation of the original variables.
To obtain the parametric solution, set the free variables $v_{1}, \ldots, v_{k}$ equal to parameters $t_{1}, \ldots, t_{k}$, where $-\infty<t_{j}<\infty, 1 \leq j \leq k$. Then solve equations (9) for the leading variables $u_{1}, \ldots, u_{m}$ and back-substitute for
$v_{1}, \ldots, v_{k}$ to obtain a Gaussian parametric solution

$$
\begin{align*}
u_{1} & =D_{1}-E_{11} t_{1}-\cdots-E_{1 k} t_{k} \\
u_{2} & =D_{2}-E_{21} t_{1}-\cdots-E_{2 k} t_{k} \\
& \vdots \\
u_{m} & =D_{m}-E_{m 1} t_{1}-\cdots-E_{m k} t_{k}  \tag{10}\\
v_{1} & =t_{1} \\
v_{2} & =t_{2} \\
& \vdots \\
v_{k} & =t_{k}
\end{align*}
$$

To illustrate, the reduced echelon system

$$
\begin{array}{ll}
x+4 w+u+v & =1 \\
y-u+v & =2  \tag{11}\\
z-w+2 u-v & =0
\end{array}
$$

has variable list $x, w, u, v, y, z$, listed in order of first appearance. The lead variables are $x, y, z$ and the free variables are $w, u, v$. Assign parameters $t_{1}, t_{2}, t_{3}$ to the free variables and back-substitute in (11) to obtain a Gaussian parametric solution

$$
\begin{aligned}
x & =1-4 t_{1}-t_{2}-t_{3} \\
y & =2+t_{2}-t_{3} \\
z & =t_{1}-2 t_{2}+t_{3}, \\
w & =t_{1}, \\
u & =t_{2} \\
v & =t_{3} .
\end{aligned}
$$

$$
\begin{aligned}
x & =1-4 t_{1}-t_{2}-t_{3} \\
w & =t_{1} \\
\text { or } & =t_{2} \\
v & =t_{3} \\
y & =2+t_{2}-t_{3} \\
z & =t_{1}-2 t_{2}+t_{3}
\end{aligned}
$$

Computers and Reduced Echelon form. Computer algebra systems and computer numerical laboratories compute from a given linear system (1) a new system of the same size, which has a special form:

## Definition 4 (Reduced Echelon form)

A linear homogeneous system (2) is in reduced echelon form, abbreviated rref, if it has the form (9) and

The leading variables $u_{1}, \ldots, u_{k}$ are re-labelings of
variable names $x_{r_{1}}, \ldots, x_{r_{k}}$ with subscripts in increasing order $r_{1}<\cdots<r_{k}$.
If $x_{i}$ is a leading variable, then variables $x_{1}, \ldots, x_{i-1}$ are absent from that equation.
(14) Any equations without variables appear last as $0=0$.

The definition for a consistent nonhomogeneous system is identical. For an inconsistent system, the definition requires a change to (14), to allow the appearance of an equation $0=1$, a false equation used primarily to detect inconsistency.

Every computer-produced reduced echelon form corresponding to a consistent system is already a reduced echelon system, except for the listing of equations without variables.

The reverse is not true. To illustrate, linear system (11) is a reduced echelon system which satisfies (12) and (13), provided the variable list is given as $x, y, z, u, v, w$.

Three rules for equivalent systems. Two systems (1) are said to be equivalent provided they have exactly the same solutions. For the purpose of solving systems, there are three reversible operations on equations which can be applied to obtain equivalent systems:

| Swap | Two equations can be interchanged without changing |
| :--- | :--- |
| the solution set. |  |
| Multiply | An equation can be multiplied by $c \neq 0$ without chang- |
|  | ing the solution set. |

Combination A multiple of one equation can be added to a different equation without changing the solution set.

The last two rules replace an existing equation by a new one. The multiply rule is reversed by multiplication by $1 / c$, whereas the combination rule is reversed by subtracting the equation-multiple previously added.

Exposition of a set of equations and its equivalent system under these rules demands that all equations be copied, not just the affected equation(s). Generally, each displayed system changes just one equation, although in certain cases, such as swapping equations, this rigor is relaxed.

Gaussian elimination. This algorithm seeks a target equivalent system of equations which is a reduced echelon system (9), by applying to each algebraic step one of the three rules.

At each stage of the algebra an equivalent system is obtained. This means that no solutions are gained or lost throughout the algebraic steps: the original and target systems have exactly the same solutions.
The target reduced echelon system (9) has an easily-found Gaussian parametric solution (10), which is reported as the general solution.

## Theorem 3 (Gaussian Elimination)

Every linear system (1) has either no solutions or else it has the same solutions as an equivalent reduced echelon system (9), obtained by applying the three rules of swap, multiply and combination.

A Gaussian Algorithm. An equation is processed if a lead variable has been selected and that variable has been eliminated from all other equations. Otherwise, the equation is unprocessed.

1. If an equation " $0=0$ " appears, then discard it. If an equation " $0=c$ " appears and $c \neq 0$, then the system is inconsistent (no solution) and the algorithm terminates.
2. Identify among the variables names $x_{1}, \ldots, x_{n}$ a lead variable $x_{r}$ in an unprocessed equation. Apply the multiply rule to insure leading coefficient one. Apply the combination rule to eliminate variable $x_{r}$ from all other equations.
3. Apply the swap rule to move this equation to the lowest position following the already-processed equations, but before the unprocessed equations. Mark the equation as processed, e.g., replace $x_{r}$ by $x_{r}$.
4. Repeat steps $1-3$, until all equations have been processed once. Then lead variables $u_{1}, \ldots, u_{m}$ have been defined and the resulting system is in reduced echelon form (9).

Uniqueness and the reduced echelon form. Unfortunately, the Gaussian algorithm performed on a given system by two different persons may result in two different reduced echelon systems. To make the answer unique, attention has to be paid to the natural order of the variable list $x_{1}, \ldots, x_{n}$. Uniqueness results by adding one more critical step to the Gaussian algorithm:

RREF. Always select a lead variable as the next possible variable name in the original list order $x_{1}, \ldots, x_{n}$, taken from all possible unprocessed equations.

This extra step insures that the reduced echelon system (9) is in reduced echelon form, that is, additionally (12), (13) are satisfied. The rref requirement (14) is ignored here, because equation " $0=0$ " is not listed and equation " $0=1$ " stops the algorithm.
The wording next possible must be used, because once a variable name is used for a lead variable it may not be used again. The next variable following the last-used lead variable, from the list $x_{1}, \ldots, x_{n}$, may not appear in any unprocessed equation, in which case it is a free variable. The next variable name in the original list order is then tried as a lead variable.

Avoiding fractions. Integer arithmetic should be used, when possible, to speed up hand computation in the Gaussian algorithm. To avoid fractions, Gaussian algorithm step 2 may be modified to read with leading coefficient nonzero. The final division to obtain leading coefficient one is then delayed until last.

Detection of no solution. The appearance at any stage of the algorithm of an equation like " $0=1$ " signals no solutions, whereupon the algorithm stops, and we report the system is inconsistent. The reason for no solution is that " $0=1$ " is false, therefore the equivalent system at that stage cannot have a solution. By equivalence, the original system cannot have a solution, either.

Basis. The terminology basis is used to refer to the $k$ homogeneous trial solutions that appear in the answer check, as applied to the Gaussian parametric solution (10). Knowledge of these solutions is enough to write out the general solution to the homogeneous system, hence the terminology basis is appropriate. The reader is warned that many different Gaussian parametric solutions are possible, for example, by re-ordering the variable list. Therefore, a basis is not unique. Language like the basis is fundamentally incorrect. The reader should verify for the example $x+y+z=0$ and Gaussian parametric solution $x=-t_{1}-t_{2}, y=t_{1}$, $z=t_{2}$ that a basis is given by the two solutions

$$
\begin{array}{lll}
x=-1, & y=1, & z=0, \\
x=-1, & y=0, & z=1 .
\end{array}
$$

Nullspace. The word space in this context has meaning taken from the phrases storage space and parking space - it has no geometrical meaning whatsoever. The term nullspace refers to the set of all solutions of the homogeneous system, identical to the set of all combinations of the basis elements in (8). The nullspace remains unchanged regardless of the choice of basis. The reader should verify for the example $x+y+z=0$ and the two Gaussian parametric solutions $x=-t_{1}-t_{2}$, $y=t_{1}, z=t_{2}$ and $x=t_{1}, y=-t_{1}-t_{2}, z=t_{2}$ that two possible bases for the nullspace are given by the equations

$$
\left\{\begin{array} { l l l } 
{ x = - 1 , } & { y = 1 , } & { z = 0 , } \\
{ x = - 1 , } & { y = 0 , } & { z = 1 . }
\end{array} \quad \text { and } \quad \left\{\begin{array}{lll}
x=1, & y=-1, & z=0, \\
x=0, & y=-1, & z=1 .
\end{array}\right.\right.
$$

Nullity and Rank. The terminology nullity applied to a system of equations refers to the number of free variables appearing in any equivalent reduced echelon system, which exactly matches the number $k$ of basis elements in relation (8). The rank of a system of equations is correspondingly the number of lead variables. The fundamental relation is

$$
\text { nullity }+ \text { rank }=\text { number of variables. }
$$

In literature, nullity is referred to as the dimension of the nullspace. The term dimension is a synonym for the number of free variables,
which is exactly the number of parameters in a minimal parametric solution for the linear system.

Unique solution. There is a unique solution to a system of equations exactly when no free variables are present. This is identical to requiring that the number $n$ of variables equal the number of lead variables, or $\operatorname{rank}=\mathbf{n}$.

Infinitely many solutions. The situation of infinitely many solutions occurs exactly when there is at least one free variable to which a parameter, say $t_{1}$, is assigned. Since this parameter takes the values $-\infty<t_{1}<\infty$, there are an infinity of solutions. The condition rank less than $\mathbf{n}$ can replace a reference to the number of free variables.
Homogeneous systems are always consistent, therefore if the number of variables exceeds the number of equations, then there is always one free variable, and this gives the following basic result of linear algebra.

## Theorem 4 (Existence of Infinitely Many Solutions)

A system of $m \times n$ linear homogeneous equations (2) with $m<n$ has an infinite number of solutions, hence it has a nonzero solution.

1 Example (Line of Intersection) Show that the parametric equations $x=$ $2-6 t, y=-1-t, z=8 t$ represent a line through $(2,-1,0)$ with tangent $-6 \vec{\imath}-\vec{\jmath}$ which is the line of intersection of the three planes

$$
\begin{align*}
x+2 y+z & =0, \\
2 x-4 y+z & =8  \tag{15}\\
3 x-2 y+2 z & =8 .
\end{align*}
$$

Solution: Using $t=0$ in the parametric solution shows that $(2,-1,0)$ is on the line. The tangent is $x^{\prime}(t) \vec{\imath}+y^{\prime}(t) \vec{\jmath}+z^{\prime}(t) \vec{k}$ which computes to $-6 \vec{\imath}-\vec{\jmath}$. The details for showing the parametric solution satisfies the three equations simultaneously:

$$
\begin{aligned}
\text { LHS } & =x+2 y+z & & \text { First equation left side. } \\
& =(2-6 t)+2(-1-t)+8 t & & \text { Substitute parametric solution. } \\
& =0 & & \text { Matches the RHS in (15). } \\
\text { LHS } & =2 x-4 y+z & & \text { Second equation left side. } \\
& =2(2-6 t)-4(-1-t)+8 t & & \text { Substitute. } \\
& =8 & & \text { Matches (15). } \\
\text { LHS } & =3 x-2 y+2 z & & \text { Third equation left side. } \\
& =3(2-6 t)-2(-1-t)+16 t & & \text { Substitute. } \\
& =8 & & \text { Matches (15). }
\end{aligned}
$$

2 Example (Reduced Echelon System) Solve the system and interpret the solution geometrically.

$$
\begin{array}{r}
y+z=1, \\
x+2 z=3 .
\end{array}
$$

Solution: The parametric solution is a line:

$$
\begin{aligned}
& x=3-2 t_{1}, \\
& y=1-t_{1}, \quad-\infty<t_{1}<\infty . \\
& z=t_{1}, \quad
\end{aligned}
$$

In standard $x y z$-coordinates, this line passes through $(3,1,0)$ with tangent direction $-2 \vec{\imath}-\vec{\jmath}+\vec{k}$.
Details. To justify this solution, a reduced echelon system (9) is identified and the easily-found Gaussian parametric solution (10) is obtained. The variable list has six possible orderings, but the order of appearance $y, z, x$ will be used in this example.

$$
\begin{array}{rlrl}
y+z & =1, & & \text { Reduced echelon system, lead variables } y, x \text { and } \\
x+2 z=3 . & & \text { free variable } z . \\
y=1-z, & & \text { Solve for lead variables } y \text { and } x \text { in terms of the } \\
x=3-2 z, & & \text { free variable } z, \text { then assign to } z \text { parameter } t_{1} . \\
z=t_{1} . & & \\
y=1-t_{1}, & & \text { Back-substitute for free varable } z . \text { This is the } \\
y=3-2 t_{1}, & & \text { Gaussian parametric solution. It is geometrically } \\
x=t_{1} . & & \text { a line, by Table } 1 .
\end{array}
$$

Answer check. The displayed answer can be checked manually by substituting the parametric solution into the equations $y+z=1, x+2 z=3$, as follows:

$$
\begin{aligned}
y+z & =\left(1-t_{1}\right)+\left(t_{1}\right) \\
& =1, \\
x+2 z & =\left(3-2 t_{1}\right)+2\left(t_{1}\right) \\
& =3 .
\end{aligned}
$$

Therefore, the two equations are satisfied for all values of $t_{1}$.

3 Example (Gaussian Elimination) Solve the system by applying the Gaussian algorithm on page 290.

$$
\begin{aligned}
w+2 x-y+z & =1, \\
w+3 x-y+2 z & =0, \\
x+z & =-1 .
\end{aligned}
$$

Solution: The answer using the natural variable list order $w, x, y, z$ is the Gaussian parametric solution

$$
\begin{aligned}
w & =3+t_{1}+t_{2}, \\
x & =-1-t_{2}, \\
y & =t_{1}, \\
z & =t_{2},
\end{aligned} \quad-\infty<t_{1}, t_{2}<\infty . \quad \begin{aligned}
& \\
& x
\end{aligned} \quad-\quad .
$$

Details. The Gaussian algorithm will be applied. The details amount to applying the three rules swap, multiply and combination for equivalent equations (page 289) to obtain a target reduced echelon system, whose Gaussian parametric solution matches the one reported above. For clarity, lead variables are are marked with an asterisk $(*)$ and processed equations are marked with lead variable enclosed in a box, e.g., $w$.


Answer check. The check will be performed according to the outline on page 286.

Step 1. The nonhomogeneous trial solution $w=3, x=-1, y=z=0$ is obtained by setting $t_{1}=t_{2}=0$. It is required to satisfy the nonhomogeneous system

$$
\begin{aligned}
w+2 x-y+z & =1, \\
w+3 x-y+2 z & =0, \\
x+z & =-1 .
\end{aligned}
$$

Step 2. The partial derivatives $\partial_{t_{1}}, \partial_{t_{2}}$ are applied to the parametric solution to obtain two homogeneous trial solutions $w=1, x=0, y=1$, $z=0$ and $w=1, x=-1, y=0, z=1$, which are required to satisfy the homogeneous system

$$
\begin{aligned}
w+2 x-y+z & =0, \\
w+3 x-y+2 z & =0, \\
x+z & =0
\end{aligned}
$$

Each trial solution from Step 1 and Step 2 is checked by direct substitution.
4 Example (No solution) Verify by applying the Gaussian algorithm that the system has no solution.

$$
\begin{aligned}
w+2 x-y+z & =0 \\
w+3 x-y+2 z & =0 \\
x+z & =1
\end{aligned}
$$

Solution: The Gaussian algorithm (page 290) will be applied, using the three rules swap, multiply and combination for equivalent equations (page 289).

$$
\begin{array}{rlrlrl}
w^{*}+2 x-y+z & =0 & & \text { Original system. Select variable or- } \\
w+3 x-y+2 z & =0 & & \begin{array}{l}
\text { der } w, x, y, z .
\end{array} \\
\text { able } w .
\end{array}
$$

The appearance of the equation " $0=1$ " signals no solution. The logic: if the original system has a solution, then so does the present equivalent system, hence $0=1$, a contradiction. The Gaussian algorithm stops, because of the inconsistent system containing the false equation " $0=1$."

5 Example (Reduced Echelon form) Find an equivalent system in reduced echelon form using the Gaussian algorithm and the RREF step (page 290 ).

$$
\begin{aligned}
x_{1}+2 x_{2}-x_{3}+x_{4} & =1, \\
x_{1}+3 x_{2}-x_{3}+2 x_{4} & =0, \\
x_{2} & +x_{4}=-1 .
\end{aligned}
$$

Solution: The answer using the natural variable list order $x_{1}, x_{2}, x_{2}, x_{4}$ is the reduced echelon system in reduced echelon form (briefly, rref form)

$$
\begin{array}{lllll}
x_{1} & -x_{3} & -x_{4}=3 \\
& & & & x_{2}
\end{array}
$$

The Gaussian parametric solution of this system is

$$
\begin{array}{ll}
x_{1}=3+t_{1}+t_{2}, & \\
x_{2}=-1-t_{2}, & \\
x_{3}=t_{1}, & -\infty<t_{1}, t_{2}<\infty . \\
x_{4}=t_{2}, &
\end{array}
$$

The details are exactly the same as Example 3, with $w=x_{1}, x=x_{2}, y=x_{3}$, $z=x_{4}$.
The answer can be checked using maple. The output below duplicates the system reported above, plus a single $0=0$ equation at the end.
with(linalg):
A: =matrix $([[1,2,-1,1,1],[1,3,-1,2,0],[0,1,0,1,-1]])$;
$R:=\operatorname{rref}(A)$; geneqns $(R,[x[1], x[2], x[3], x[4],-1])$;

Proof of Theorem 1: The proof will follow from the lemma and theorem below.

Lemma 1 (Unique Representation) If a set of parametric equations (4) satisfies (5), (6) and (7), then each solution of linear system (1) is given by (4) for exactly one set of parameter values.

Proof: Let a solution of system (1) be given by (4) for two sets of parameters $t_{1}, \ldots, t_{k}$ and $\bar{t}_{1}, \ldots, \bar{t}_{k}$. By (7), $t_{j}=x_{i_{j}}=\bar{t}_{j}$ for $1 \leq j \leq k$, therefore the parameter values are the same.

## Theorem 5 (Minimal Parametric Solutions)

Let linear system (1) have a parametric solution satisfying (4), (5), (6). Then (4) has the fewest possible parameters if and only if each solution of linear system (1) is given by (4) for exactly one set of parameter values.

Proof: Suppose first that a general solution (4) is given with the least number $k$ of parameters, but contrary to the theorem, there are two ways to represent some solution, with corresponding parameters $r_{1}, \ldots, r_{k}$ and also $s_{1}, \ldots, s_{k}$. Subtract the two sets of parametric equations, thus eliminating the symbols $x_{1}$, $\ldots, x_{n}$, to obtain:

$$
\begin{array}{rc}
c_{11}\left(r_{1}-s_{1}\right)+\cdots+c_{1 k}\left(r_{k}-s_{k}\right) & =0 \\
& \vdots \\
c_{n 1}\left(r_{1}-s_{1}\right)+\cdots+c_{n k}\left(r_{k}-s_{k}\right) & =0 .
\end{array}
$$

Relabel the variables and constants so that $r_{1}-s_{1} \neq 0$, possible since the two sets of parameters are supposed to be different. Divide the preceding equations by $r_{1}-s_{1}$ and solve for the constants $c_{11}, \ldots, c_{n 1}$. This results in equations

$$
\begin{aligned}
c_{11} & =c_{12} w_{2}+\cdots+c_{1 k} w_{k} \\
& \vdots \\
c_{n 1} & =c_{n 2} w_{2}+\cdots+c_{n k} w_{k},
\end{aligned}
$$

where $w_{j}=-\frac{r_{j}-s_{j}}{r_{1}-s_{1}}, 2 \leq j \leq k$. Insert these relations into (4), effectively eliminating the symbols $c_{11}, \ldots, c_{n 1}$, to obtain

$$
\begin{aligned}
x_{1} & =d_{1}+c_{12}\left(t_{2}+w_{2} t_{1}\right)+\cdots+c_{1 k}\left(t_{k}+w_{k} t_{1}\right) \\
x_{2} & =d_{2}+c_{22}\left(t_{2}+w_{2} t_{1}\right)+\cdots+c_{2 k}\left(t_{k}+w_{k} t_{1}\right), \\
& \vdots \\
x_{n} & =d_{n}+c_{n 2}\left(t_{2}+w_{2} t_{1}\right)+\cdots+c_{n k}\left(t_{k}+w_{k} t_{1}\right) .
\end{aligned}
$$

Let $t_{1}=0$. The remaining parameters $t_{2}, \ldots, t_{k}$ are fewer parameters that describe all solutions of the system, a contradiction to the definition of $k$. This completes the proof of the first half of the theorem.

To prove the second half of the theorem, assume that a parametric solution (4) is given which represents all possible solutions of the system and in addition each solution is represented by exactly one set of parameter values. It will be established that the number $k$ in (4) is the least possible parameter count.
Suppose not. Then there is a second parametric solution

$$
\begin{align*}
x_{1} & =e_{1}+b_{11} v_{1}+\cdots+b_{1 \ell} v_{\ell}, \\
& \vdots  \tag{16}\\
x_{n} & =e_{n}+b_{n 1} v_{1}+\cdots+b_{n \ell} v_{\ell},
\end{align*}
$$

where $\ell<k$ and $v_{1}, \ldots, v_{\ell}$ are the parameters. It is assumed that (16) represents all solutions of the linear system.
We shall prove that the solutions for zero parameters in (4) and (16) can be taken to be the same, that is, another parametric solution is given by

$$
\begin{align*}
x_{1} & =d_{1}+b_{11} s_{1}+\cdots+b_{1 \ell} s_{\ell}, \\
& \vdots  \tag{17}\\
x_{n} & =d_{n}+b_{n 1} s_{1}+\cdots+b_{n \ell} s_{\ell} .
\end{align*}
$$

The idea of the proof is to substitute $x_{1}=d_{1}, \ldots, x_{n}=d_{n}$ into (16) for parameters $r_{1}, \ldots, r_{n}$. Then solve for $e_{1}, \ldots, e_{n}$ and replace back into (16) to obtain

$$
\begin{aligned}
x_{1} & =d_{1}+b_{11}\left(v_{1}-r_{1}\right)+\cdots+b_{1 \ell}\left(v_{\ell}-r_{\ell}\right), \\
& \vdots \\
x_{n} & =d_{n}+b_{n 1}\left(v_{1}-r_{1}\right)+\cdots+b_{n \ell}\left(v_{\ell}-r_{\ell}\right) .
\end{aligned}
$$

Replacing parameters $s_{j}=v_{j}-r_{j}$ gives (17).
From (4) it is known that $x_{1}=d_{1}+c_{11}, \ldots, x_{n}=d_{n}+c_{n 1}$ is a solution. By (17), there are constants $r_{1}, \ldots, r_{\ell}$ such that (we cancel $d_{1}, \ldots, d_{n}$ from both sides)

$$
\begin{aligned}
c_{11} & =b_{11} r_{1}+\cdots+b_{1 \ell} r_{\ell}, \\
& \vdots \\
c_{n 1} & =b_{n 1} r_{1}+\cdots+b_{n \ell} r_{\ell} .
\end{aligned}
$$

If $r_{1}$ through $r_{\ell}$ are all zero, then the solution just referenced equals $d_{1}, \ldots$, $d_{n}$, hence (4) has a solution that can be represented with parameters all zero or with $t_{1}=1$ and all other parameters zero, a contradiction. Therefore, some $r_{i} \neq 0$ and we can assume by renumbering that $r_{1} \neq 0$. Return now to the last system of equations and divide by $r_{1}$ in order to solve for the constants $b_{11}$, $\ldots, b_{n 1}$. Substitute the answers back into (17) in order to obtain parametric equations

$$
\begin{aligned}
x_{1} & =d_{1}+c_{11} w_{1}+b_{12} w_{2}+\cdots+b_{1 \ell} w_{\ell}, \\
& \vdots \\
x_{n} & =d_{n}+c_{n 1} w_{1}+b_{n 2} w_{2}+\cdots+b_{n \ell} w_{\ell},
\end{aligned}
$$

where $w_{1}=s_{1}, w_{j}=s_{j}-r_{j} / r_{1}$. Given $s_{1}, \ldots, s_{\ell}$ are parameters, then so are $w_{1}, \ldots, w_{\ell}$.
This process can be repeated for the solution $x_{1}=d_{1}+c_{12}, \ldots, x_{n}=d_{n}+c_{n 2}$. We assert that for some index $j, 2 \leq j \leq \ell$, constants $b_{i j}, \ldots, b_{n j}$ in the previous display can be isolated, and the process of replacing symbols $b$ by $c$ continued.

If not, then $w_{2}=\cdots=w_{\ell}=0$. Then solution $x_{1}, \ldots, x_{n}$ has two distinct representations in (4), first with $t_{2}=1$ and all other $t_{j}=0$, then with $t_{1}=w_{1}$ and all other $t_{j}=0$. A contradiction results, which proves the assertion. After $\ell$ repetitions of this replacement process, we find a parametric solution

$$
\begin{aligned}
x_{1} & =d_{1}+c_{11} u_{1}+c_{12} u_{2}+\cdots+c_{1 \ell} u_{\ell}, \\
& \vdots \\
x_{n} & =d_{n}+c_{n 1} u_{1}+c_{n 2} u_{2}+\cdots+c_{n \ell} u_{\ell},
\end{aligned}
$$

in some set of parameters $u_{1}, \ldots, u_{\ell}$.
However, $\ell<k$, so at least the solution $x_{1}=d_{1}+c_{1 k}, \ldots, x_{n}=d_{n}+c_{n k}$ remains unused by the process. Insert this solution into the previous display, valid for some parameters $u_{1}, \ldots, u_{\ell}$. The relation says that the solution $x_{1}=d_{1}, \ldots$, $x_{n}=d_{n}$ in (4) has two distinct sets of parameters, namely $t_{1}=u_{1}, \ldots, t_{\ell}=u_{\ell}$, $t_{k}=-1$, all others zero, and also all parameters zero, a contradiction. This completes the proof of the theorem.

## Exercises 8.1

Planar System. Solve the $x y$-system and interpret the solution geometrically.

1. $\begin{aligned} x+y & =1, \\ y & =1 .\end{aligned}$
2. $\begin{aligned} x+y & =-1, \\ x & =3 .\end{aligned}$
3. $x+y=1$,
$x+2 y=2$.
4. $x+y=1$,
$x+2 y=3$.
5. $x+y=1$,
$2 x+2 y=2$.
6. $2 x+y=1$,
7. $x-y=1$,

$$
-x-y=-1
$$

8. $2 x-y=1$,
$x-0.5 y=0.5$.
9. $\begin{aligned} & x=1, \\ & x=2\end{aligned}$
10. $\quad-y=1, ~ 子 \quad-y=2 . ~$
11. $x+y=1$,

$$
x+y=2
$$

12. $x-y=1$,

$$
x-y=0
$$

Reduced Echelon System. Solve the $x y z$-system and interpret the solution geometrically.
13. $y+z=1$,
$x+2 z=2$.
14. $x+z=1$,
$y+2 z=4$.
15. $y+z=1$,
$x+3 z=2$.
16. $x+z=1$,
$y+z=5$.
17. $x+z=1$,
$2 x+2 z=2$.
18. $x+y=1$,
$3 x+3 y=3$.
19. $x+y+z=1$.
20. $x+2 y+4 z=0$.

Homogeneous System. Solve the $x y z$-system using the Gaussian algorithm and variable order $x, y, z$.
21. $\begin{aligned} & y+z=0, \\ & 2 x+2 z=0, \\ & x+z=0, \\ & \text { 22. } \\ & x+y+z=0, \\ & x+y-z=0, \\ & x+y+z=0, \\ & x+y+z=0, \\ & \text { 23. } \quad 2 x+2 z=0, \\ & x+z=0 . \\ & x+y+z=0, \\ & \text { 24. } \\ & 2 x+2 z=0, \\ & 3 x+y+3 z=0 .\end{aligned}$

Nonhomogeneous System. Solve the system using the Gaussian algorithm and variable order $x, y, z$.
25. $\begin{aligned} y+z & =1, \\ 2 x+2 z & =2, \\ x+z & =1 .\end{aligned}$
$2 x+y+z=1$,
26. $x+2 z=2$,
$x+y-z=-1$.

$$
x+y+z=0,
$$

27. $2 x+2 z=0$,
$x+z=0$.
$x+y+z=0$,
28. $2 x+2 z=0$, $3 x+y+3 z=0$.

Nonhomogeneous System. Solve the system using the Gaussian algorithm and variable order $y, z, u, v$.

$$
\begin{aligned}
& y+z+4 u+8 v=10, \\
& \text { 29. } 2 z-u+v=10 \text {, } \\
& 2 y-u+5 v=10 \text {. } \\
& y+z+4 u+8 v=10, \\
& \text { 30. } 2 z-2 u+2 v=0 \text {, } \\
& y+3 z+2 u+5 v=5 \text {. }
\end{aligned}
$$

31 $y+z+4 u+8 v=1$,
$y+3 z+2 u+6 v=1$.
$y+3 z+4 u+8 v=1$,
32. $2 z-2 u+4 v=0$,
$y+3 z+2 u+6 v=1$.

Nullspace. Solve using the Gaussian algorithm and variable order $y, z$, $u, v$. Report the values of the nullity and rank in the equation nullity + rank $=4$.
33. $2 z-u+v=0$,

$$
y+z+4 u+8 v=0,
$$

$2 y-u+5 v=0$.
34. $2 z-2 u+2 v=0$,

$$
y+z+4 u+8 v=0,
$$

$y-z+6 u+6 v=0$.
$y+z+4 u+8 v=0$,
35. $2 z-2 u+4 v=0$,

$$
y+3 z+2 u+6 v=0 .
$$

$y+3 z+4 u+8 v=0$,
36. $2 z-2 u+4 v=0$,
$y+3 z+2 u+12 v=0$.
RREF. For each $3 \times 5$ homogeneous system, (a) solve using the Gaussian algorithm and variable order $x, y, z, u$, $v$, (b) Report an equivalent set of equations in reduced echelon form (rref).

$$
x+y+z+4 u+8 v=0,
$$

37. $-x+2 z-2 u+2 v=0$,
$y-z+6 u+6 v=0$.

$$
x+y+z+4 u+8 v=0,
$$

38. $-2 z-u+v=0$,

$$
2 y \quad-u+5 v=0 \text {. }
$$

$$
y+z+4 u+8 v=0
$$

39. $x+2 z-2 u+4 v=0$, $2 x+y+3 z+2 u+6 v=0$.
$x+y+3 z+4 u+8 v=0$,
40. $2 x+2 z-2 u+4 v=0$,

$$
x-y+3 z+2 u+12 v=0 \text {. }
$$

