Unforced Oscillations

- Simple Harmonic Motion
  - Hooke's Law
  - Newton's Second Law
  - Method of Force Competition
- Visualization of Harmonic Motion
- Phase-Amplitude Conversion
- The Simple Pendulum and The Linearized Pendulum
- The Physical Pendulum
- The Swinging Rod
- Torsional Pendulum: Mechanical Wristwatch
- Shockless Auto
- Rolling Wheel on a Spring

Simple Harmonic Motion

Consider the spring-mass system of Figure 1, where x measures the signed distance from the equilibrium position of the mass. The spring is assumed to exert a force under both compression and elongation. Such springs are commonly used in automotive suspension systems, notably coil springs and leaf springs. In the case of coil springs, it is assumed that there is space between the coils, which allows the spring to exert bidirectional forces.

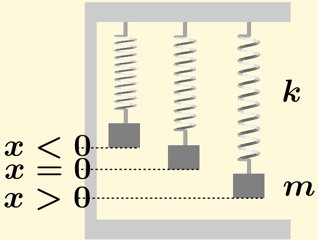

Figure 1. An undamped spring-mass system, showing compression, equilibrium and elongation of the spring with corresponding positions of the mass m.

Hooke's law

> The linear restoring force F exerted by a spring is proportional to the signed elongation X, briefly, $F = -kX$.

The number k is called **Hooke's constant** for the spring. In the model of Figure 1, $X = x(t)$ and $k > 0$. The minus sign accounts for the action of the force: the spring tries to **restore** the mass to the equilibrium state, so the vector force is directed toward the equilibrium position $x = 0$.

Newton's Second Law

Specialized to the model in Figure 1, Newton's second law says:

> The force F exerted by a mass m attached to a spring is $F = ma$ where $a = d^2x/dt^2$ is the acceleration of the mass.

The **weight** $W = mg$ is defined in terms of the **gravitational constant** $g = 32$ ft/s^2, 9.8 m/s^2 or 980 cm/s^2 where the mass m is given respectively in **slugs**, **kilograms** or **grams**. The weight is the force due to gravity and it has the appropriate units for a force: **pounds** in the case of the fps system of units.

Method of Force Competition

Hooke's law $F = -kx(t)$ and Newton's second law $F = mx''(t)$ give two independent equations for the force acting on the system. Equating competing forces implies that the signed displacement $x(t)$ satisfies the **free vibration** equation

$$mx''(t) + kx(t) = 0.$$

It is also called the **harmonic oscillator**, especially in its equivalent form

$$x''(t) + \omega^2 x(t) = 0, \quad \omega^2 = \frac{k}{m}.$$

In this context, ω is the **natural frequency** of the free vibration. The harmonic oscillator is said to describe a **simple harmonic motion** $x(t)$, known by Euler's constant-coefficient *recipe* to have the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Visualization of Harmonic Motion

A simple harmonic motion can be obtained graphically by means of the experiment shown in Figure 2, in which an undamped spring-mass system has an attached pen that writes on a moving paper chart. The chart produces the simple harmonic motion

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

or equivalently

$$x(t) = A \cos(\omega t - \phi).$$

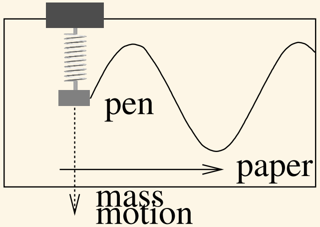

Figure 2. A moving paper chart records the vertical motion of a mass on a spring by means of an attached pen.

Phase-Amplitude Conversion

Given a simple harmonic motion $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$, as in Figure 3, define **amplitude** A and **phase angle** α by the formulas

$$A = \sqrt{c_1^2 + c_2^2}, \quad c_1 = A\cos\alpha \quad \text{and} \quad c_2 = A\sin\alpha.$$

Then the simple harmonic motion has the **phase-amplitude form**

$$x(t) = A\cos(\omega t - \alpha). \tag{1}$$

To directly obtain (1) from trigonometry, use the trigonometric identity $\cos(a-b) = \cos a\cos b + \sin a \sin b$ with $a = \omega t$ and $b = \alpha$. It is known from trigonometry that $x(t)$ has **period** $2\pi/\omega$ and **phase shift** α/ω. A full period is called a **cycle** and a half-period a **semicycle**. The **frequency** $\omega/(2\pi)$ is the number of complete cycles per second, or the reciprocal of the period.

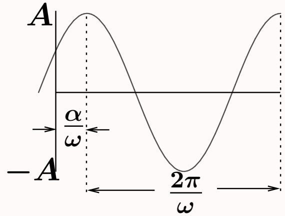

Figure 3. Simple harmonic oscillation $x(t) = A\ \cos(\omega t - \alpha)$, showing the period $2\pi/\omega$, the phase shift α/ω and the amplitude A.

The Simple Pendulum

A pendulum is constructed from a thin massless wire or rod of length L and a body of mass m, as in Figure 4. Along the circular arc traveled by the mass, the velocity is ds/dt where $s = L\theta(t)$ is arclength. The acceleration is $L\theta''(t)$. Newton's second law for the force along this arc is $F = mL\theta''(t)$. Another relation for the force can be found by resolving the vector gravitational force $m\vec{g}$ into its normal and tangential components. By trigonometry, the tangential component gives a second force equation $F = -mg\sin\theta(t)$. Equating competing forces and canceling m results in the **pendulum equation**

$$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0. \tag{2}$$

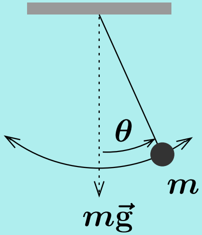

Figure 4. A simple pendulum

The Linearized Pendulum

The approximation $\sin u \approx u$, valid for small angles u, is applied to the pendulum equation

$$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0.$$

The result is the **linearized pendulum**

$$\theta''(t) + \frac{g}{L}\theta(t) = 0. \tag{3}$$

This equation is indistinguishable from the classical harmonic oscillator, except for variable names. The characteristic solution is

$$\theta(t) = A\cos(\omega t - \alpha), \qquad \omega^2 = g/L.$$

The Physical Pendulum

The **compound pendulum** or **physical pendulum** is a rigid body of total mass m having center of mass C which is suspended from a fixed origin O – see Figure 5.

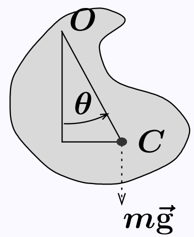

Figure 5. The physical pendulum

Derivation. The distance from O to C is denoted $d > 0$. The gravity vector $\vec{g}$ makes angle θ with segment $\overline{OC}$ and it supplies a restoring torque of magnitude $F = -mgd \sin\theta$. Newton's second law gives a second force equation $F = I\theta''(t)$ where I is the torque of the rigid body about O. Force competition results in the **compound pendulum**

(4) $$\theta''(t) + \frac{mgd}{I}\sin\theta(t) = 0.$$

Using $\sin u \approx u$ gives the **linearized compound pendulum**

(5) $$\theta''(t) + \omega^2\theta(t) = 0, \qquad \omega = \sqrt{\frac{mgd}{I}}.$$

The Swinging Rod

As depicted in Figure 6, a swinging rod is a special case of the compound pendulum.

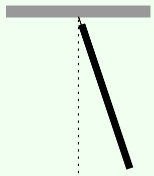

Figure 6. The swinging rod

Derivation. The torque $I = mL^2/3$ and center of mass distance $d = L/2$ are routine calculus calculations. Then $mgd/I = 3mgL/2mL^2 = 3g/2L$. Then use models (4) and (5).

Applying (4) gives the **swinging rod model**

$$\theta''(t) + \frac{3g}{2L}\sin\theta(t) = 0 \tag{6}$$

and applying (5) gives the **linearized swinging rod model**

$$\theta''(t) + \omega^2\theta(t) = 0, \qquad \omega = \sqrt{\frac{3g}{2L}}. \tag{7}$$

The Torsional Pendulum

A model for a balance wheel in a watch, a gavanometer or a Cavendish torsional balance is the torsional pendulum, which is a rigid body suspended by a wire – see Figure 7. The twisted wire exerts a restoring force $F = -\kappa\theta_0$ when the body is rotated through angle θ_0. There is no small angle restriction on this restoring force, because it acts in the spirit of Hooke's law like a linear spring restoring force. The model uses the Newton's second law force relation $F = I\theta_0''(t)$, as in the physical pendulum, but the restoring force is $F = -\kappa\theta_0$, giving the **torsional pendulum**

$$\theta_0''(t) + \omega^2\theta_0(t) = 0, \quad \omega = \sqrt{\frac{\kappa}{I}}. \tag{8}$$

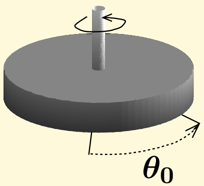

Figure 7. The torsional pendulum, a model for a balance wheel in a watch. The wheel rotates angle θ_0 about the vertical axis, which acts as a spring, exerting torque I against the rotation.

Shockless Auto

An auto loaded with several occupants is supported by four coil springs, as in Figure 8, but all of the shock absorbers are worn out. The simplistic model $mx''(t) + kx(t) = 0$ will be applied. The plan is to estimate the number of seconds it takes for one complete oscillation. This is the time between two consecutive *bottom–outs* of the auto.

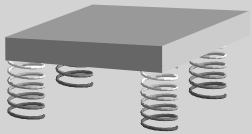

Figure 8. A model for a car on four springs

Application

Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter. The load is divided among the four springs equally, so each spring supports $m = 1350/4$ Kg. We will find the natural frequency of vibration ω. Then the number of seconds for one complete oscillation is the period $T = 2\pi/\omega$ seconds. The oscillation model for one spring is

$$\frac{1350}{4}x''(t) + 20000x(t) = 0.$$

In the harmonic oscillator form $x'' + \omega^2 x = 0$, $\omega^2 = \frac{20000(4)}{1350} = 59.26$ and therefore $\omega = 7.70$, $T = 2\pi/\omega = 0.82$ seconds.

Rolling Wheel on a Spring

A wheel of total mass m and radius R is attached at its center to a spring of Hooke's constant k, as in Figure 9. The wheel rolls without slipping.

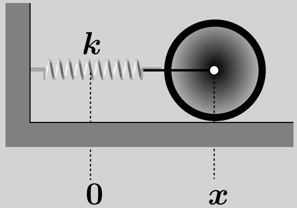

Figure 9. A rolling wheel attached to a spring.

Modeling. Let $x(t)$ be the elongation of the spring from equilibrium, $x > 0$ corresponding to the wheel rolling to the right and $x < 0$ corresponding to the wheel rolling to the left.

If the wheel slides frictionless, then the model is $mx''(t) + kx(t) = 0$. But a wheel that rolls without slipping has inertia, and consideration of this physical difference will be shown to give the **rolling wheel** equation

$$mx''(t) + \frac{2}{3}kx(t) = 0. \tag{9}$$

Derivation of the Rolling Wheel Model

The derivation is based upon the energy conservation law

$$\text{Kinetic} + \text{Potential} = \text{constant}.$$

The kinetic energy T is the sum of two energies, $T_1 = \frac{1}{2}mv^2$ for translation and $T_2 = \frac{1}{2}I\omega^2$ for the rolling wheel, whose inertia is $I = \frac{1}{2}mR^2$. The velocity is $v = R\omega = x'(t)$. Algebra gives $T = T_1 + T_2 = \frac{3}{4}mv^2$. The potential energy is $K = \frac{1}{2}kx^2$ for a spring of Hooke's constant k. Application of the energy conservation law $T + K = c$ gives the equation $\frac{3}{4}m(x'(t))^2 + \frac{1}{2}k(x(t))^2 = c$. Differentiate this equation on t to obtain $\frac{3}{2}mx'(t)x''(t) + kx(t)x'(t) = 0$, then cancel $x'(t)$ to give

$$mx''(t) + \frac{2}{3}kx(t) = 0.$$