Applications of Systems of Differential Equations

- Brine Tank Cascade
- Cascade Model
- Recycled Brine Tank Cascade
- Recycled Cascade Model

Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 20, 40, 60, respectively, as in Figure 1.

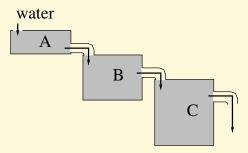


Figure 1. Three brine tanks in cascade.

Assumptions and Notation

- It is supposed that fluid enters tank A at rate r, drains from A to B at rate r, drains from B to C at rate r, then drains from tank C at rate r. Hence the volumes of the tanks remain constant. Let r = 10, to illustrate the ideas.
- Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.
- Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. We suppose added to tank A water containing no salt. Therefore, the salt in all the tanks is eventually lost from the drains.

Cascade Model

The cascade is modeled by the **chemical balance law**

rate of change = input rate - output rate.

Application of the balance law results in the triangular differential system

$$x_1' = -rac{1}{2}x_1, \ x_2' = rac{1}{2}x_1 - rac{1}{4}x_2, \ x_3' = rac{1}{4}x_2 - rac{1}{6}x_3.$$

Cascade Model Solution

The solution is justified by the integrating factor method for first order scalar differential equations.

$$egin{aligned} x_1(t) &= x_1(0)e^{-t/2}, \ x_2(t) &= -2x_1(0)e^{-t/2} + (x_2(0) + 2x_1(0))e^{-t/4}, \ x_3(t) &= rac{3}{2}x_1(0)e^{-t/2} - 3(x_2(0) + 2x_1(0))e^{-t/4} \ &+ (x_3(0) - rac{3}{2}x_1(0) + 3(x_2(0) + 2x_1(0)))e^{-t/6}. \end{aligned}$$

Recycled Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 60, 30, 60, respectively, as in Figure 2.

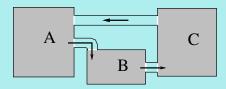


Figure 2. Three brine tanks in cascade with recycling.

Assumptions and Notation

- Suppose that fluid drains from tank A to B at rate r, drains from tank B to C at rate r, then drains from tank C to A at rate r. The tank volumes remain constant due to constant recycling of fluid. For purposes of illustration, let r = 10.
- Uniform stirring of each tank is assumed, which implies uniform salt concentration throughout each tank.
- Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. No salt is lost from the system, due to recycling.

Recycled Cascade Model

Using compartment analysis, the recycled cascade is modeled by the non-triangular system

$$egin{array}{lll} x_1' &=& -rac{1}{6}x_1 & + rac{1}{6}x_3, \ x_2' &=& rac{1}{6}x_1 - rac{1}{3}x_2, \ x_3' &=& rac{1}{3}x_2 - rac{1}{6}x_3. \end{array}$$

Recycled Cascade Solution

$$egin{aligned} x_1(t) &= c_1 + (c_2 - 2c_3)e^{-t/3}\cos(t/6) + (2c_2 + c_3)e^{-t/3}\sin(t/6), \ x_2(t) &= rac{1}{2}c_1 + (-2c_2 - c_3)e^{-t/3}\cos(t/6) + (c_2 - 2c_3)e^{-t/3}\sin(t/6), \ x_3(t) &= c_1 + (c_2 + 3c_3)e^{-t/3}\cos(t/6) + (-3c_2 + c_3)e^{-t/3}\sin(t/6). \end{aligned}$$

- At infinity, $x_1 = x_3 = c_1$, $x_2 = c_1/2$. The meaning is that the total amount of salt is uniformly distributed in the tanks, in the ratio 2:1:2.
- The solution of the system was found by the eigenanalysis method. It can also be found by the resolvent method in Laplace theory.