Piecewise Representation of Switching Inputs

- Unit step.
- Pulse function.
- Ramp function.
- Piecewise-defined functions.
- Laplace Pulse Example.
- Laplace of a piecewise-defined function.
Unit Step Function

Definition:

\[ \text{step}(t - a) = \begin{cases} 
1 & t \geq a, \\
0 & t < a. 
\end{cases} \]

Example: Write the piecewise defined function \( f(t) \) in terms of unit step functions.

\[ f(t) = \begin{cases} 
\sin t & t \geq \pi, \\
0 & t < \pi,
\end{cases} \]

Solution:

\[ f(t) = \sin(t) \begin{cases} 
1 & t \geq \pi, \\
0 & t < \pi.
\end{cases} = \sin(t) \text{ step}(t - \pi). \]
Pulse Function

Definition:

\[
pulse(t, a, b) = \begin{cases} 
1 & a \leq t < b, \\
0 & t < a, t \geq b, 
\end{cases} 
= \text{step}(t - a) - \text{step}(t - b).
\]

Example: Write the piecewise defined function \( f(t) \) in terms of pulse functions.

\[
f(t) = \begin{cases} 
\sin t & 0 \leq t < \pi, \\
\cos t & \pi \leq t < 2\pi, \\
0 & \text{else.}
\end{cases}
\]

Solution:

\[
f(t) = \sin(t) \begin{cases} 
1 & 0 \leq t < \pi \\
0 & \text{else}
\end{cases} + \cos(t) \begin{cases} 
1 & \pi \leq t < 2\pi, \\
0 & \text{else}
\end{cases} 
= \sin(t) \pulse(t, 0, \pi) + \cos(t) \pulse(t, \pi, 2\pi).
\]
Ramp Function

Definition:
\[
\text{ramp}(t - a) = \begin{cases} 
    t - a & t \geq a, \\
    0 & t < a,
\end{cases} = (t - a) \text{ step}(t - a).
\]

Example: Write the piecewise defined function \( f(t) \) in terms of ramp functions.

\[
f(t) = \begin{cases} 
    2t - 2 & t \geq 1, \\
    0 & t < 1.
\end{cases}
\]

Solution:

\[
f(t) = (2t - 2) \begin{cases} 
    1 & t \geq 1, \\
    0 & t < 1.
\end{cases} = (2t - 2) \text{ step}(t - 1) = 2 \text{ ramp}(t - 1).
\]
Piecewise Defined Functions

Definition:

\[
f(t) = \begin{cases} 
    f_1(t) & a_1 \leq t < a_2, \\
    f_2(t) & a_2 \leq t < a_3, \\
    \vdots & \vdots \\
    f_n(t) & a_n \leq t < a_{n+1}, \\
    0 & \text{else}
\end{cases}
\]

Problem: Write the piecewise defined function \( f(t) \) in terms of pulse functions.

Solution:

\[
f(t) = f_1(t) \begin{cases} 
    1 & a_1 \leq t < a_2, \\
    0 & \text{else}
\end{cases} + \cdots + f_n(t) \begin{cases} 
    1 & a_n \leq t < a_{n+1}, \\
    0 & \text{else}
\end{cases} = f_1(t) \text{pulse}(t, a_1, a_2) + \cdots + f_n(t) \text{pulse}(t, a_n, a_{n+1}).
\]
Laplace Pulse Example

\[
f(t) = \begin{cases} 
  e^{-t} & 1 \leq t < 2, \\
  \cos \pi t & 2 \leq t < 3, \\
  0 & \text{else}
\end{cases}
\]

Problem: Find \( L(f(t)) \) by pulse decomposition.

Solution: We use \( L(g(t) \text{ step}(t, a)) = e^{-as} L(g(t + a)) \).

\[
L(f(t)) = L(e^{-t} \text{ pulse}(t, 1, 2) + \cos \pi t \text{ pulse}(t, 2, 3)) \\
= L(e^{-t} \text{ step}(t, 1) - e^{-t} \text{ step}(t, 2) + \cos \pi t \text{ step}(t, 2)) - \\
L(\cos \pi t \text{ step}(t, 3)) \\
= e^{-s} L(e^{-t-1}) - e^{-2s} L(e^{-t-2}) + e^{-2s} L(\cos(\pi t + 2\pi)) - \\
e^{-3s} L(\cos(\pi t + 3\pi)) \\
= e^{-1-s} - e^{-2-2s} + \frac{se^{-2s} - se^{-3s}}{s^2 + \pi^2}.
\]
Laplace of a Piecewise Defined Function

**Definition:**

\[
f(t) = \begin{cases} 
  f_1(t) & a_1 \leq t < a_2, \\
  f_2(t) & a_2 \leq t < a_3, \\
  \vdots & \vdots \\
  f_n(t) & a_n \leq t < a_{n+1}, \\
  0 & \text{else}
\end{cases}
\]

**Problem:** Find \( L(f(t)) \) for the piecewise defined function \( f(t) \).

**Solution:** We use \( L(g(t) \ \text{step}(t, a)) = e^{-as}L(g(t + a)) \).

\[
L(f(t)) = L(f_1(t) \ \text{pulse}(t, a_1, a_2)) + \cdots + L(f_n(t) \ \text{pulse}(t, a_n, a_{n+1})) \\
= \sum_{j=1}^{n} L(f_j(t) \ \text{step}(t, a_j)) - L(f_j(t) \ \text{step}(t, a_{j+1})) \\
= \sum_{j=1}^{n} e^{-a_j s}L(f_j(t + a_j)) - e^{-a_{j+1} s}L(f_j(t + a_{j+1})).
\]