Basic Theory of Linear Differential Equations

- Picard-Lindelöf Existence-Uniqueness
  - Vector $n$th Order Theorem
  - Second Order Linear Theorem
  - Higher Order Linear Theorem

- Homogeneous Structure

- Recipe for Constant-Coefficient Linear Homogeneous Differential Equations
  - First Order
  - Second Order
  - $n$th Order

- Superposition

- Non-Homogeneous Structure
Theorem 1 (Picard-Lindelöf Existence-Uniqueness)
Let the $n$-vector function $f(x, y)$ be continuous for real $x$ satisfying $|x - x_0| \leq a$ and for all vectors $y$ in $\mathbb{R}^n$ satisfying $\|y - y_0\| \leq b$. Additionally, assume that $\partial f / \partial y$ is continuous on this domain. Then the initial value problem

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y(x)$ defined on $|x - x_0| \leq h$, satisfying $\|y - y_0\| \leq b$, for some constant $h$, $0 < h < a$.

The unique solution can be written in terms of the Picard Iterates

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt, \quad y_0(x) \equiv y_0,$$

as the formula

$$y(x) = y_n(x) + R_n(x), \quad \lim_{n \to \infty} R_n(x) = 0.$$

The formula means $y(x)$ can be computed as the iterate $y_n(x)$ for large $n$. 
Theorem 2 (Second Order Linear Picard-Lindelöf Existence-Uniqueness)

Let the coefficients $a(x), b(x), c(x), f(x)$ be continuous on an interval $J$ containing $x = x_0$. Assume $a(x) \neq 0$ on $J$. Let $g_1$ and $g_2$ be real constants. The initial value problem

\[
\begin{align*}
    a(x)y'' + b(x)y' + c(x)y &= f(x), \\
    y(x_0) &= g_1, \\
    y'(x_0) &= g_2
\end{align*}
\]

has a unique solution $y(x)$ defined on $J$. 
Theorem 3 (Higher Order Linear Picard-Lindelöf Existence-Uniqueness)
Let the coefficients $a_0(x), \ldots, a_n(x), f(x)$ be continuous on an interval $J$ containing $x = x_0$. Assume $a_n(x) \neq 0$ on $J$. Let $g_1, \ldots, g_n$ be constants. Then the initial value problem

$$
\begin{cases}
  a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = f(x), \\
  y(x_0) = g_1, \\
  y'(x_0) = g_2, \\
  \vdots \\
  y^{(n-1)}(x_0) = g_n
\end{cases}
$$

has a unique solution $y(x)$ defined on $J$. 
Theorem 4 (Homogeneous Structure 2nd Order)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where $c_1, c_2$ are arbitrary constants and $y_1(x), y_2(x)$ are independent solutions.

Theorem 5 (Homogeneous Structure n-th Order)
The homogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x),$$

where $c_1, \ldots, c_n$ are arbitrary constants and $y_1(x), \ldots, y_n(x)$ are independent solutions.
Theorem 6 (First Order Recipe)
Let $a$ and $b$ be constants, $a \neq 0$. Let $r_1$ denote the root of $ar + b = 0$ and construct its corresponding atom $e^{r_1x}$. Multiply the atom by arbitrary constant $c_1$. Then $y = c_1e^{r_1x}$ is the general solution of the first order equation

$$ay' + by = 0.$$ 

The equation $ar + b = 0$, called the characteristic equation, is found by the formal replacements $y' \rightarrow r, y \rightarrow 1$ in the differential equation $ay' + by = 0$. 
Theorem 7 (Second Order Recipe)
Let $a \neq 0$, $b$ and $c$ be real constant. Then the general solution of

$$ay'' + by' + cy = 0$$

is given by the expression $y = c_1y_1 + c_2y_2$, where $c_1$, $c_2$ are arbitrary constants and $y_1, y_2$ are two atoms constructed as outlined below from the roots of the characteristic equation

$$ar^2 + br + c = 0.$$
Construction of Atoms for Second Order

The atom construction from the roots \( r_1, r_2 \) of \( ar^2 + br + c = 0 \) is based on Euler’s theorem below, organized by the sign of the discriminant \( D = b^2 - 4ac \).

\[
\begin{align*}
D > 0 \text{ (Real distinct roots } r_1 \neq r_2) & \quad y_1 = e^{r_1x}, \quad y_2 = e^{r_2x}. \\
D = 0 \text{ (Real equal roots } r_1 = r_2) & \quad y_1 = e^{r_1x}, \quad y_2 = xe^{r_1x}. \\
D < 0 \text{ (Conjugate roots } r_1 = \bar{r}_2 = A + iB) & \quad y_1 = e^{Ax} \cos(Bx), \\
& \quad y_2 = e^{Ax} \sin(Bx).
\end{align*}
\]

Theorem 8 (Euler’s Theorem)
The atom \( y = x^k e^{Ax} \cos(Bx) \) is a solution of \( ay'' + by' + cy = 0 \) if and only if \( r_1 = A + iB \) is a root of the characteristic equation \( ar^2 + br + c = 0 \) and \( (r - r_1)^k \) divides \( ar^2 + br + c \).

Valid also for \( \sin(Bx) \) when \( B > 0 \). Always, \( B \geq 0 \). For second order, only \( k = 1, 2 \) are possible.

Euler’s theorem is valid for any order differential equation: replace the equation by \( a_n y^{(n)} + \cdots + a_0 y = 0 \) and the characteristic equation by \( a_n r^n + \cdots + a_0 = 0 \).
**Theorem 9 (Recipe for $n$th Order)**

Let $a_n \neq 0, \ldots, a_0$ be real constants. Let $y_1, \ldots, y_n$ be the list of $n$ distinct atoms constructed by Euler's Theorem from the $n$ roots of the characteristic equation

$$a_n r^n + \cdots + a_0 = 0.$$ 

Then $y_1, \ldots, y_n$ are independent solutions of

$$a_n y^{(n)} + \cdots + a_0 y = 0$$

and all solutions are given by the general solution formula

$$y = c_1 y_1 + \cdots + c_n y_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

The characteristic equation is found by the formal replacements $y^{(n)} \rightarrow r^n, \ldots, y' \rightarrow r, y \rightarrow 1$ in the differential equation.
Theorem 10 (Superposition)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has the superposition property:

If $y_1$, $y_2$ are solutions and $c_1$, $c_2$ are constants, then the combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is a solution.

The result implies that linear combinations of solutions are also solutions.

The theorem applies as well to an $n$th order linear homogeneous differential equation with continuous coefficients $a_0(x), \ldots, a_n(x)$.

The result can be extended to more than two solutions. If $y_1, \ldots, y_k$ are solutions of the differential equation, then all linear combinations of these solutions are also solutions.

The solution space of a linear homogeneous $n$th order linear differential equation is a subspace $S$ of the vector space $V$ of all functions on the common domain $J$ of continuity of the coefficients.
**Theorem 11 (Non-Homogeneous Structure 2nd Order)**

The non-homogeneous equation \( a(x)y'' + b(x)y' + c(x)y = f(x) \) has general solution

\[
y(x) = y_h(x) + y_p(x),
\]

where

- \( y_h(x) \) is the general solution of the homogeneous equation \( a(x)y'' + b(x)y' + c(x)y = 0 \), and
- \( y_p(x) \) is a particular solution of the nonhomogeneous equation \( a(x)y'' + b(x)y' + c(x)y = f(x) \).

The theorem is valid for higher order equations: the general solution of the non-homogeneous equation is \( y = y_h + y_p \), where \( y_h \) is the general solution of the homogeneous equation and \( y_p \) is any particular solution of the non-homogeneous equation.

**An Example**

For equation \( y'' - y = 10 \), the homogeneous equation \( y'' - y = 0 \) has general solution \( y_h = c_1 e^x + c_2 e^{-x} \). Select \( y_p = -10 \), an equilibrium solution. Then \( y = y_h + y_p = c_1 e^x + c_2 e^{-x} - 10 \). 
Theorem 12 (Non-Homogeneous Structure $n$th Order)
The non-homogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)$ has general solution
\[ y(x) = y_h(x) + y_p(x), \]
where
- $y_h(x)$ is the general solution of the homogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$, and
- $y_p(x)$ is a particular solution of the nonhomogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)$. 