Systems of Differential Equations and Laplace’s Method

- Solving $x' = Cx$
- The Resolvent
- An Illustration for $x' = Cx$
Solving $x' = Cx$.

Apply $L$ to each side to obtain $L(x') = CL(x)$. Use the parts rule

$$L(x') = sL(x) - x(0)$$

to obtain

$$sL(x) - x(0) = L(Cx)$$
$$sL(x) - L(Cx) = x(0)$$
$$sI L(x) - C L(x) = x(0)$$
$$(sI - C)L(x) = x(0).$$

Resolvent

The inverse of $sI - C$ is called the **resolvent**, a term invented to describe the equation

$$L(x(t)) = (sI - C)^{-1}x(0).$$
An Illustration for $x' = Cx$

Define $C = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which gives a scalar initial value problem

$$
\begin{align*}
  x_1'(t) &= 2x_1(t) + 3x_2(t), \\
  x_2'(t) &= 4x_2(t), \\
  x_1(0) &= 1, \\
  x_2(0) &= 2.
\end{align*}
$$

Then the adjugate formula $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$ gives the resolvent

$$
(sI - C)^{-1} = \frac{1}{(s - 2)(s - 4)} \begin{pmatrix} s - 4 & 3 \\ 0 & s - 2 \end{pmatrix}.
$$
The Laplace transform of the solution is then

\[ L(x(t)) = (sI - C)^{-1} \left( \begin{array}{c} 1 \\ 2 \end{array} \right) = \begin{pmatrix} \frac{s + 2}{(s - 2)(s - 4)} \\ \frac{2}{s - 4} \end{pmatrix}. \]

Partial fractions and use of the backward Laplace table imply

\[ x(t) = \begin{pmatrix} 3e^{4t} - 2e^{2t} \\ 2e^{4t} \end{pmatrix}. \]