Second Order Systems

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Coupled Spring-Mass Systems

Three masses are attached to each other by four springs as in Figure 1. A model will be developed for the positions of the three masses.

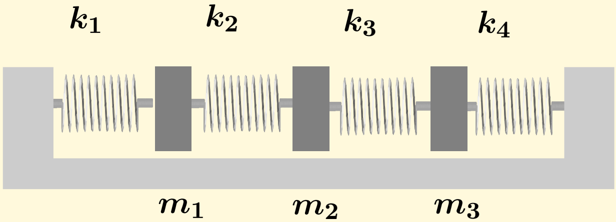

Figure 1. Three masses connected by springs. The masses slide along a frictionless horizontal surface.

Variables

The analysis uses the following constants, variables and assumptions.

Mass Constants	The masses m_1, m_2, m_3 are assumed to be point masses concentrated at their center of gravity.
Spring Constants	The mass of each spring is negligible. The springs operate according to Hooke's law: Force = k(elongation). Constants k_1, k_2, k_3, k_4 denote the Hooke's constants. The springs restore after compression and extension.
Position Variables	The symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the mass positions along the horizontal surface, measured from their equilibrium positions, plus right and minus left.
Fixed Ends	The first and last spring are attached to fixed walls.

Derivation

The **competition method** is used to derive the equations of motion. In this case, the law is

Newton's Second Law Force = Sum of the Hooke's Forces.

The model equations are

$$
\begin{aligned}
m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\
m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\
m_3 x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4 x_3(t).
\end{aligned} \tag{1}
$$

- The equations are justified in the case of all positive variables by observing that the first three springs are elongated by x_1, $x_2 - x_1$, $x_3 - x_2$, respectively. The last spring is compressed by x_3, which accounts for the minus sign.
- Another way to justify the equations is through mirror-image symmetry: interchange $k_1 \longleftrightarrow k_4$, $k_2 \longleftrightarrow k_3$, $x_1 \longleftrightarrow x_3$, then equation 2 should be unchanged and equation 3 should become equation 1.

Vector-Matrix form $M\mathbf{x}'' = K\mathbf{x}$

In vector-matrix form, this system is a **second order system**

$$M\mathbf{x}''(t) = K\mathbf{x}(t)$$

where the **displacement** $\mathbf{x}$, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

$$K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Vector-Matrix form $\mathbf{x}'' = \boldsymbol{A}\mathbf{x}$

Because M is invertible, the system can always be re-written using $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{K}$ as the second-order system

$$\mathbf{x}'' = \boldsymbol{A}\mathbf{x}.$$

Details

$$\begin{aligned} \boldsymbol{M}\mathbf{x}'' &= \boldsymbol{K}\mathbf{x} \\ \boldsymbol{M}^{-1}\boldsymbol{M}\mathbf{x}'' &= \boldsymbol{M}^{-1}\boldsymbol{K}\mathbf{x} \\ \mathbf{x}'' &= \boldsymbol{A}\mathbf{x} \end{aligned}$$

Laplace Operations on $x'' = Ax$

Apply $\boldsymbol{L}$ to each side to obtain $\boldsymbol{L}(\mathrm{x}'') = \boldsymbol{A}\boldsymbol{L}(\mathrm{x})$. Use the vector parts rule $\boldsymbol{L}(\mathrm{x}') = \boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \mathrm{x}(0)$ to obtain

$$\begin{array}{lcl} \boldsymbol{L}(\mathrm{x}'') & = & \boldsymbol{A}\boldsymbol{L}(\mathrm{x}) \\ \boldsymbol{s}\boldsymbol{L}(\mathrm{x}') - \mathrm{x}'(0) & = & \boldsymbol{A}\boldsymbol{L}(\mathrm{x}) \\ \boldsymbol{s}(\boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \mathrm{x}(0)) - \mathrm{x}'(0) & = & \boldsymbol{A}\boldsymbol{L}(\mathrm{x}) \\ \boldsymbol{s}^2\boldsymbol{L}(\mathrm{x}) - \boldsymbol{s}\mathrm{x}(0) - \mathrm{x}'(0) & = & \boldsymbol{A}\boldsymbol{L}(\mathrm{x}) \\ (\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{L}(\mathrm{x}) & = & \boldsymbol{s}\mathrm{x}(0) + \mathrm{x}'(0) \\ \boldsymbol{L}(\mathrm{x}(\boldsymbol{t})) & = & (\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}\mathrm{x}(0) + \mathrm{x}'(0)) \,. \end{array}$$

The inverse of $\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{C}$ is also called a **resolvent**, and we have the **Laplace Resolvent Formula**

$$\boldsymbol{L}(\mathrm{x}(\boldsymbol{t})) = (\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}\mathrm{x}(0) + \mathrm{x}'(0)) \,.$$

Formal Solution of $\mathbf{x}'' = A\mathbf{x}$

- The solution of a second order vector-matrix equation $\mathbf{x}'' = A\mathbf{x}$ can be formally written because of Laplace resolvent theory as the formula

$$\mathbf{x}(t) = L^{-1}\left((s^2 I - A)^{-1}(s\mathbf{x}(0) + \mathbf{x}'(0))\right).$$

- The Cayley-Hamilton Method, applied to the related first order system $\mathbf{u}' = C\mathbf{u}$, implies that the components of $\mathbf{u}$, which are exactly the components of $\mathbf{x}$ and $\mathbf{x}'$, are vector linear combinations of the $2n$ atoms obtained from the roots of $\det(C - rI) = 0$. Laplace theory will deliver the precise atoms that must be used. For now, we summarize:

$$\mathbf{x}(t) = (\text{atom}_1)\vec{\mathbf{c}}_1 + \cdots + (\text{atom}_{2n})\vec{\mathbf{c}}_{2n}$$

System $x'' = Ax$ when A has eigenvalues $\lambda \leq 0$

We'll assume that each eigenvalue λ of A has multiplicity one, for simplicity. Eigenvalues $\det(\lambda I - A) = 0$, which implies the resolvent

$$\left(s^2 I - A\right)^{-1} = \frac{\textbf{adj}(s^2 I - A)}{\det(s^2 I - A)}$$

has denominator roots $s^2 = \lambda$ = negative number or zero.

- Nonzero roots s have the form

$$s = \pm\omega i, \quad \omega = \sqrt{-\lambda}.$$

- A zero eigenvalue $\lambda = 0$ corresponds to a double root $s = 0$.

Solution Structure for $\mathrm{x}'' = A\mathrm{x}$

Assume the eigenvalues of A are 0, $-\omega_1^2, \ldots, -\omega_k^2$. Equivalently,

$$\det(s^2 I - A) = 0$$

has a double root $s = 0$, and complex conjugate roots $s = \pm\omega_\ell i$, $1 \le \ell \le k$. We'll assume the eigenvalues of A have multiplicity one, for simplicity.

Laplace partial fraction theory applied to denominator factors $(s^2 + \omega_\ell^2)$ or s^2 implies the solution x is a vector linear combination of $2n = 2k + 2$ atoms

$$1, t, \cos\omega_1 t, \sin\omega_1 t, \ldots, \cos\omega_k t, \sin\omega_k t,$$

If zero is not an eigenvalue of A, then factor s^2 does not appear in the denominator, which causes the first two atoms 1, t to be removed. Then $k = n$ and all atoms involve sines and cosines.

A Break from Mathematics

Pierre Laplace, eighteenth century mathematician and astronomer, states in his major treatise on *Celestial Mechanics*

> "A very few fundamental laws can explain an extraordinary number of very complex phenomena."

Composers and mathematicians differ in their instruments, but they hear the same music, experience beauty equally, and each seeks the same clarity and explanation. They are not especially physicians, psychiatrists, lawyers, politicians nor clergy.

But these other groups have identified their own set of fundamental laws, and they use them to describe an extraordinary number of complexities.