

## Transform Properties

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Collected here are the major theorems for the manipulation of Laplace transform tables.

- Lerch's Cancellation Law
- Linearity
- The Parts Rule ( $t$ -Derivative Rule)
- The  $t$ -Integral Rule
- The  $s$ -Differentiation Rule
- First Shifting Rule
- Second Shifting Rule
- Periodic Function Rule
- Convolution Rule

### Theorem 1 (Lerch)

If  $f_1(t)$  and  $f_2(t)$  are continuous, of exponential order and

$$\int_0^{\infty} f_1(t)e^{-st} dt = \int_0^{\infty} f_2(t)e^{-st} dt$$

for all  $s > s_0$ , then for  $t \geq 0$ ,

$$f_1(t) = f_2(t).$$

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The result is remembered as the cancelation law

$$L(f_1(t)) = L(f_2(t)) \text{ implies } f_1(t) = f_2(t).$$

## **Theorem 2 (Linearity)**

The Laplace transform has these inherited integral properties:

- (a)  $L(f(t) + g(t)) = L(f(t)) + L(g(t)),$
- (b)  $L(cf(t)) = cL(f(t)).$

### Theorem 3 (The Parts Rule)

Let  $y(t)$  be continuous, of exponential order and let  $y'(t)$  be piecewise continuous on  $t \geq 0$ . Then  $L(y'(t))$  exists and

$$L(y'(t)) = sL(y(t)) - y(0).$$

### Theorem 4 (The $t$ -Integral Rule)

Let  $g(t)$  be of exponential order and continuous for  $t \geq 0$ . Then

$$L\left(\int_0^t g(x) dx\right) = \frac{1}{s}L(g(t)).$$

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- The parts rule is also called the  $t$ -derivative rule. It is used to remove derivatives  $y'$  from Laplace equations.
  - The two rules are related by  $y(t) = \int_0^t g(x) dx$ .

### **Theorem 5 (The $s$ -Differentiation Rule)**

Let  $f(t)$  be of exponential order. Then

$$L(tf(t)) = -\frac{d}{ds}L(f(t)).$$

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The rule says that each factor of  $(t)$  in the integrand of a Laplace integral can be crossed out provided an operation  $-d/ds$  is inserted in front of the integral. It is remembered as

*multiplying by  $(-t)$  differentiates the transform.*

### **Theorem 6 (First Shifting Rule)**

Let  $f(t)$  be of exponential order and  $-\infty < a < \infty$ . Then

$$L(e^{at} f(t)) = L(f(t))|_{s \rightarrow (s-a)}.$$

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The rule says that an exponential factor  $e^{at}$  in the integrand can be crossed out, provided this action is compensated by replacing  $s$  by  $s - a$  in the answer. It is remembered as

*multiplying by  $e^{at}$  shifts the transform  $s \rightarrow s - a$ .*

## Heaviside Step

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The Step function is defined by  $\text{step}(t) = 1$  for  $t \geq 0$  and  $\text{step}(t) = 0$  for  $t < 0$ . It is the same as the **unit step**  $u(t)$  and the **Heaviside function**  $H(t)$ . Then  $\text{step}(t - a)$  is the step function shifted from the origin to location  $t = a$ ,

$$\text{step}(t - a) = \begin{cases} 1 & a \leq t < \infty, \\ \text{otherwise.} & \end{cases}$$

The function **pulse** is a finite interval step function defined by

$$\begin{aligned} \text{pulse}(t, a, b) &= \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} \\ &= \text{step}(t - a) - \text{step}(t - b). \end{aligned}$$

## Maple Worksheet Definitions

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step := unapply(piecewise(t >= 0, 1, 0), t);  
pulse := unapply(step(t-a) - step(t-b), (t, a, b));
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## Step Function Shifting Rule

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### Theorem 7 (Second Shifting Rule)

Let  $f(t)$  and  $g(t)$  be of exponential order and assume  $a \geq 0$ . Let  $u(t) = \text{step}(t)$ . Then

- (a)  $L(f(t - a)u(t - a)) = e^{-as}L(f(t)),$
- (b)  $L(g(t)u(t - a)) = e^{-as}L(g(t + a)).$

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The relations are used to manipulate Laplace equations that arise in differential equations with piecewise defined inputs. Electrical engineering has many such examples.



### **Theorem 8 (Periodic Function Rule)**

Let  $f(t)$  be of exponential order and satisfy  $f(t + P) = f(t)$ . Then

$$L(f(t)) = \frac{\int_0^P f(t)e^{-st} dt}{1 - e^{-Ps}}.$$

## Some Engineering Functions

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Tabulated here are common periodic functions used in engineering applications.

$$L(\mathbf{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$$

Staircase function,  
 $\mathbf{floor}(x) = \text{greatest integer } \leq x$ .

$$L(\mathbf{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$$

Square wave,  
 $\mathbf{sqw}(x) = (-1)^{\mathbf{floor}(x)}$ .

$$L(a \mathbf{trw}(t/a)) = \frac{1}{s^2} \tanh(as/2)$$

Triangular wave,  
 $\mathbf{trw}(x) = \int_0^x \mathbf{sqw}(r) dr$ .

### Theorem 9 (Convolution Rule)

Let  $f(t)$  and  $g(t)$  be of exponential order. Then

$$L(f(t))L(g(t)) = L\left(\int_0^t f(x)g(t-x)dx\right).$$

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An example:

$$\begin{aligned}\frac{1}{s^2} \frac{1}{s-2} &= L(t)L(e^{2t}) \\ &= L\left(\int_0^t xe^{2(t-x)}dx\right) \\ &= L\left(\frac{1}{3}e^{2t} - \frac{1}{2}t - \frac{1}{4}\right)\end{aligned}$$