

Basic Laplace Theory

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Laplace Integral

The integral

$$\int_0^{\infty} g(t)e^{-st} dt$$

is called the **Laplace integral** of the function $g(t)$. It is defined by

$$\int_0^{\infty} g(t)e^{-st} dt \equiv \lim_{N \rightarrow \infty} \int_0^N g(t)e^{-st} dt$$

and it depends on variable s . The ideas will be illustrated for $g(t) = 1$, $g(t) = t$ and $g(t) = t^2$. Results appear in Table 1 *infra*.

A Basic LaPlace Table

$$\begin{aligned}\int_0^{\infty} (1)e^{-st} dt &= -(1/s)e^{-st} \Big|_{t=0}^{t=\infty} \\ &= 1/s\end{aligned}$$

Laplace integral of $g(t) = 1$.

Assumed $s > 0$.

$$\begin{aligned}\int_0^{\infty} (t)e^{-st} dt &= \int_0^{\infty} -\frac{d}{ds}(e^{-st}) dt \\ &= -\frac{d}{ds} \int_0^{\infty} (1)e^{-st} dt \\ &= -\frac{d}{ds}(1/s) \\ &= 1/s^2\end{aligned}$$

Laplace integral of $g(t) = t$.

Use

$$\int \frac{d}{ds} F(t, s) dt = \frac{d}{ds} \int F(t, s) dt.$$

Use $L(1) = 1/s$.

Differentiate.

$$\begin{aligned}\int_0^{\infty} (t^2)e^{-st} dt &= \int_0^{\infty} -\frac{d}{ds}(te^{-st}) dt \\ &= -\frac{d}{ds} \int_0^{\infty} (t)e^{-st} dt \\ &= -\frac{d}{ds}(1/s^2) \\ &= 2/s^3\end{aligned}$$

Laplace integral of $g(t) = t^2$.

Use $L(t) = 1/s^2$.

Summary

Table 1. Laplace integral $\int_0^\infty g(t)e^{-st} dt$ for $g(t) = 1, t$ and t^2 .

$$\int_0^\infty (1)e^{-st} dt = \frac{1}{s}, \quad \int_0^\infty (t)e^{-st} dt = \frac{1}{s^2}, \quad \int_0^\infty (t^2)e^{-st} dt = \frac{2}{s^3}.$$

In summary, $L(t^n) = \frac{n!}{s^{1+n}}$

A Laplace Table for Daily Use

Solving differential equations by Laplace methods requires keeping a smallest table of Laplace integrals available, usually memorized. The last three entries will be verified later.

Table 2. A minimal Laplace integral table with L -notation

$$\int_0^{\infty} (t^n) e^{-st} dt = \frac{n!}{s^{1+n}}$$

$$\int_0^{\infty} (e^{at}) e^{-st} dt = \frac{1}{s-a}$$

$$\int_0^{\infty} (\cos bt) e^{-st} dt = \frac{s}{s^2 + b^2}$$

$$\int_0^{\infty} (\sin bt) e^{-st} dt = \frac{b}{s^2 + b^2}$$

$$L(t^n) = \frac{n!}{s^{1+n}}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(\cos bt) = \frac{s}{s^2 + b^2}$$

$$L(\sin bt) = \frac{b}{s^2 + b^2}$$

Laplace Integral

The **Laplace integral** or the **direct Laplace transform** of a function $f(t)$ defined for $0 \leq t < \infty$ is the ordinary calculus integration problem

$$\int_0^{\infty} f(t)e^{-st} dt.$$

The *Laplace integrator* is $dx = e^{-st} dt$ instead of the usual dt .

A Laplace integral is succinctly denoted in science and engineering literature by the symbol

$$L(f(t)),$$

which abbreviates

$$\int_E (f(t)) dx,$$

with set $E = [0, \infty)$ and Laplace integrator $dx = e^{-st} dt$.

Some Transform Rules

$$L(f(t) + g(t)) = L(f(t)) + L(g(t))$$

The integral of a sum is the sum of the integrals.

$$L(cf(t)) = cL(f(t))$$

Constants c pass through the integral sign.

$$L(y'(t)) = sL(y(t)) - y(0)$$

The t -derivative rule, or integration by parts.

Lerch's Cancellation Law and the Fundamental Theorem of Calculus _____

$L(y(t)) = L(f(t))$ implies $y(t) = f(t)$ Lerch's cancellation law.

Lerch's cancellation law in integral form is

$$(1) \quad \int_0^{\infty} y(t)e^{-st} dt = \int_0^{\infty} f(t)e^{-st} dt \quad \text{implies} \quad y(t) = f(t).$$

Quadrature Methods _____

Lerch's Theorem is used *last* in Laplace's quadrature method. In Newton calculus, the quadrature method uses the Fundamental Theorem of Calculus *first*. The two theorems have a similar use, to *isolate* the solution y of the differential equation.

An illustration

Laplace's method will be applied to solve the initial value problem

$$y' = -1, \quad y(0) = 0.$$

Illustration Details

Table 3. Laplace method details for $y' = -1$, $y(0) = 0$.

$$y'(t)e^{-st} dt = -e^{-st} dt$$

Multiply $y' = -1$ by $e^{-st} dt$.

$$\int_0^{\infty} y'(t)e^{-st} dt = \int_0^{\infty} -e^{-st} dt$$

Integrate $t = 0$ to $t = \infty$.

$$\int_0^{\infty} y'(t)e^{-st} dt = -1/s$$

Use Table 1.

$$s \int_0^{\infty} y(t)e^{-st} dt - y(0) = -1/s$$

Integrate by parts on the left.

$$\int_0^{\infty} y(t)e^{-st} dt = -1/s^2$$

Use $y(0) = 0$ and divide.

$$\int_0^{\infty} y(t)e^{-st} dt = \int_0^{\infty} (-t)e^{-st} dt$$

Use Table 1.

$$y(t) = -t$$

Apply Lerch's cancellation law.

Translation to L -notation

Table 4. Laplace method L -notation details for $y' = -1$, $y(0) = 0$ translated from Table 3.

$$L(y'(t)) = L(-1)$$

Apply L across $y' = -1$, or multiply $y' = -1$ by $e^{-st} dt$, integrate $t = 0$ to $t = \infty$.

$$L(y'(t)) = -1/s$$

Use Table 1 forwards.

$$sL(y(t)) - y(0) = -1/s$$

Integrate by parts on the left.

$$L(y(t)) = -1/s^2$$

Use $y(0) = 0$ and divide.

$$L(y(t)) = L(-t)$$

Apply Table 1 backwards.

$$y(t) = -t$$

Invoke Lerch's cancelation law.

1 Example (Laplace method) Solve by Laplace's method the initial value problem $y' = 5 - 2t$, $y(0) = 1$ to obtain $y(t) = 1 + 5t - t^2$.

Solution: Laplace's method is outlined in Tables 3 and 4. The L -notation of Table 4 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$$L(y'(t)) = L(5 - 2t)$$

Apply L across $y' = 5 - 2t$.

$$= 5L(1) - 2L(t)$$

Linearity of the transform.

$$= \frac{5}{s} - \frac{2}{s^2}$$

Use Table 1 forwards.

$$sL(y(t)) - y(0) = \frac{5}{s} - \frac{2}{s^2}$$

Apply the t -derivative rule.

$$L(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3}$$

Use $y(0) = 1$ and divide.

$$L(y(t)) = L(1) + 5L(t) - L(t^2)$$

Use Table 1 backwards.

$$= L(1 + 5t - t^2)$$

Linearity of the transform.

$$y(t) = 1 + 5t - t^2$$

Invoke Lerch's cancelation law.

2 Example (Laplace method) Solve by Laplace's method the initial value problem $y'' = 10$, $y(0) = y'(0) = 0$ to obtain $y(t) = 5t^2$.

Solution: The L -notation of Table 4 will be used to find the solution $y(t) = 5t^2$.

$$L(y''(t)) = L(10)$$

$$sL(y'(t)) - y'(0) = L(10)$$

$$s[sL(y(t)) - y(0)] - y'(0) = L(10)$$

$$s^2L(y(t)) = 10L(1)$$

$$L(y(t)) = \frac{10}{s^3}$$

$$L(y(t)) = L(5t^2)$$

$$y(t) = 5t^2$$

Apply L across $y'' = 10$.

Apply the t -derivative rule to y' .

Repeat the t -derivative rule, on y .

Use $y(0) = y'(0) = 0$.

Use Table 1 forwards. Then divide.

Use Table 1 backwards.

Invoke Lerch's cancelation law.