Determinant Theory

- Unique Solution of $Ax = b$
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**Unique Solution of a $2 \times 2$ System**

The $2 \times 2$ system

\[
\begin{align*}
ax + by &= e, \\
 cx + dy &= f,
\end{align*}
\]

has a unique solution provided $\Delta = ad - bc$ is nonzero, in which case the solution is given by

\[
\begin{align*}
x &= \frac{de - bf}{ad - bc}, \\
y &= \frac{af - ce}{ad - bc}.
\end{align*}
\]

This result is called **Cramer’s Rule** for $2 \times 2$ systems, learned in college algebra.
Determinants of Order 2

College algebra introduces matrix notation and determinant notation:

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}. \]

Evaluation of \( \text{det}(A) \) is by Sarrus’ \( 2 \times 2 \) Rule:

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \]

The first product \( ad \) is the product of the main diagonal entries and the other product \( bc \) is from the anti-diagonal.

Cramer’s \( 2 \times 2 \) rule in determinant notation is

\[ (3) \quad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \]
Relation to Inverse Matrices

System

\[
\begin{align*}
ax + by &= e, \\
rx + dy &= f,
\end{align*}
\]

can be expressed as the vector-matrix system \( Au = b \) where

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} e \\ f \end{pmatrix}.
\]

Inverse matrix theory implies

\[
A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad u = A^{-1}b = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.
\]

Cramer’s Rule is a compact summary of the unique solution of system (4).
Unique Solution of an $n \times n$ System

System

\begin{align*}
a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\
a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\
\vdots &\quad \vdots \quad \cdots \quad \vdots \quad \vdots \\
a_{n1}x_1 &+ a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{align*}

\begin{equation}
(5)
\end{equation}

can be written as an $n \times n$ vector-matrix equation $A\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. The system has a unique solution provided the determinant of coefficients $\Delta = \det(A)$ is nonzero, and then Cramer’s Rule for $n \times n$ systems gives

\begin{equation}
(6)
x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad x_n = \frac{\Delta_n}{\Delta}.
\end{equation}

Symbol $\Delta_j = \det(B)$, where matrix $B$ has the same columns as matrix $A$, except $\col(B, j) = \vec{b}$. 

Determinants of Order $n$ ____________________________________________

Determinants will be defined shortly; intuition from the $2 \times 2$ case and Sarrus’ rule should suffice for the moment.
Determinant Notation for Cramer’s Rule

The determinant of coefficients for system $A\vec{x} = \vec{b}$ is denoted by

$$\Delta = \begin{vmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}.$$  \hspace{1cm} (7)

The other $n$ determinants in Cramer’s rule (6) are given by

$$\Delta_1 = \begin{vmatrix}
  b_1 & a_{12} & \cdots & a_{1n} \\
  b_2 & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_n & a_{n2} & \cdots & a_{nn}
\end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix}
  a_{11} & a_{12} & \cdots & b_1 \\
  a_{21} & a_{22} & \cdots & b_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & b_n
\end{vmatrix}.$$  \hspace{1cm} (8)
Given an \( n \times n \) matrix \( A \), define

\[
\text{det}(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}.
\]

In the formula, \( a_{ij} \) denotes the element in row \( i \) and column \( j \) of the matrix \( A \). The symbol \( \sigma = (\sigma_1, \ldots, \sigma_n) \) stands for a rearrangement of the subscripts \( 1, 2, \ldots, n \) and \( S_n \) is the set of all possible rearrangements. The nonnegative integer \( \text{parity}(\sigma) \) is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers \( \sigma_1, \ldots, \sigma_n \) into natural order \( 1, \ldots, n \).
College Algebra Definition and Sarrus’ Rule

For a $3 \times 3$ matrix, the College Algebra formula reduces to Sarrus’ $3 \times 3$ Rule

$$\det(A) = \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. $$

(10)
The number \( \det(A) \), in the \( 3 \times 3 \) case, can be computed by the algorithm in Figure 1, which parallels the one for \( 2 \times 2 \) matrices. The \( 5 \times 3 \) array is made by copying the first two rows of \( A \) into rows 4 and 5.

**Warning:** there is no Sarrus’ rule diagram for \( 4 \times 4 \) or larger matrices!

\[
\begin{align*}
\begin{array}{ccc}
\ a_{11} & a_{12} & a_{13} \\
\ a_{21} & a_{22} & a_{23} \\
\ a_{31} & a_{32} & a_{33} \\
\ a_{11} & a_{12} & a_{13} \\
\ a_{21} & a_{22} & a_{23}
\end{array}
\end{align*}
\]

\( d \in [a_1, a_2, a_3] \) and \( f \in [a_1, a_2, a_3] \)

\( e \in [a_1, a_2, a_3] \) and \( b \in [a_1, a_2, a_3] \)

\( c \in [a_1, a_2, a_3] \) and \( a \in [a_1, a_2, a_3] \)

**Figure 1.** Sarrus’ rule diagram for \( 3 \times 3 \) matrices, which gives

\[
\det(A) = (a + b + c) - (d + e + f).
\]
**Transpose Rule**

A consequence of the college algebra definition of determinant is the relation

\[ \det(A) = \det(A^T) \]

where \( A^T \) means the transpose of \( A \), obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.
How to Compute the Value of any Determinant

- **Four Rules.** These are the *Triangular Rule, Combination Rule, Multiply Rule* and the *Swap Rule*.
- **Special Rules.** These apply to evaluate a determinant as zero.
- **Cofactor Expansion.** This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.
- **Hybrid Method.** The four rules and the cofactor expansion are combined.
Four Rules

**Triangular**
The value of \( \det(A) \) for either an upper triangular or a lower triangular matrix \( A \) is the product of the diagonal elements:

\[
\det(A) = a_{11}a_{22} \cdots a_{nn}.
\]

This is a one-arrow Sarrus’ rule.

**Swap**
If \( B \) results from \( A \) by swapping two rows, then

\[
\det(A) = (-1) \det(B).
\]

**Combination**
The value of \( \det(A) \) is unchanged by adding a multiple of a row to a different row.

**Multiply**
If one row of \( A \) is multiplied by constant \( c \) to create matrix \( B \), then

\[
\det(B) = c \det(A).
\]
1 Example (Four Properties) Apply the four properties of a determinant to justify the formula

$$\det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} = 24.$$
**Solution:** Let $D$ denote the value of the determinant. Then

\[
D = \det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} \quad \text{Given.}
\]

\[
D = \det \begin{pmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix} \quad \text{combo}(1,2,-1), \quad \text{combo}(1,3,-1). \text{ Combination leaves the determinant unchanged.}
\]

\[
D = 6 \det \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix} \quad \text{Multiply rule } m = 1/6 \text{ on row 1 factors out a 6.}
\]

\[
D = 6 \det \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix} \quad \text{combo}(1,3,1), \quad \text{combo}(2,1,2).
\]

\[
D = -6 \det \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix} \quad \text{swap}(1,2). \text{ Swap changes the sign of the determinant.}
\]

\[
D = 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix} \quad \text{Multiply rule } m = -1 \text{ on row 1.}
\]

\[
D = 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix} \quad \text{combo}(2,3,-3).
\]

\[
D = 6(1)(-1)(-4) = 24 \quad \text{Triangular rule. Formula verified.}
\]
Elementary Matrices and the Four Rules

The four rules can be stated in terms of elementary matrices as follows.

**Triangular**

The value of $\det(A)$ for either an upper triangular or a lower triangular matrix $A$ is the product of the diagonal elements: $\det(A) = a_{11}a_{22}\cdots a_{nn}$. This is a one-arrow Sarrus’ rule valid for dimension $n$.

**Swap**

If $E$ is an elementary matrix for a swap rule, then $\det(EA) = (-1)\det(A)$.

**Combination**

If $E$ is an elementary matrix for a combination rule, then $\det(EA) = \det(A)$.

**Multiply**

If $E$ is an elementary matrix for a multiply rule with multiplier $m \neq 0$, then $\det(EA) = m\det(A)$.

Because $\det(E) = 1$ for a combination rule, $\det(E) = -1$ for a swap rule and $\det(E) = c$ for a multiply rule with multiplier $c \neq 0$, it follows that for any elementary matrix $E$ there is the determinant multiplication rule

$$\det(EA) = \det(E)\det(A).$$
Special Determinant Rules

The results are stated for rows but also hold for columns, because \( \det(A) = \det(A^T) \).

- **Zero row**: If one row of \( A \) is zero, then \( \det(A) = 0 \).
- **Duplicate rows**: If two rows of \( A \) are identical, then \( \det(A) = 0 \).
- **RREF \( \neq I \)**: If \( \text{rref}(A) \neq I \), then \( \det(A) = 0 \).
- **Common factor**: The relation \( \det(A) = c \det(B) \) holds, provided \( A \) and \( B \) differ only in one row, say row \( j \), for which \( \text{row}(A,j) = c \text{row}(B,j) \).
- **Row linearity**: The relation \( \det(A) = \det(B) + \det(C) \) holds, provided \( A, B \) and \( C \) differ only in one row, say row \( j \), for which \( \text{row}(A,j) = \text{row}(B,j) + \text{row}(C,j) \).
Cofactor Expansion for $3 \times 3$ Matrices

This is a review the college algebra topic, where the dimension of $A$ is $3$. **Cofactor row expansion** means the following formulas are valid:

$$
|A| = \left| \begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array} \right|
$$

$$
= a_{11}(+1) \left| \begin{array}{cc}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{array} \right| + a_{12}(-1) \left| \begin{array}{cc}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{array} \right| + a_{13}(+1) \left| \begin{array}{cc}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{array} \right|
$$

$$
= a_{21}(-1) \left| \begin{array}{cc}
a_{12} & a_{13} \\
a_{32} & a_{33}
\end{array} \right| + a_{22}(+1) \left| \begin{array}{cc}
a_{11} & a_{13} \\
a_{31} & a_{33}
\end{array} \right| + a_{23}(-1) \left| \begin{array}{cc}
a_{11} & a_{12} \\
a_{31} & a_{32}
\end{array} \right|
$$

$$
= a_{31}(+1) \left| \begin{array}{cc}
a_{12} & a_{13} \\
a_{22} & a_{23}
\end{array} \right| + a_{32}(-1) \left| \begin{array}{cc}
a_{11} & a_{13} \\
a_{21} & a_{23}
\end{array} \right| + a_{33}(+1) \left| \begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array} \right|
$$

The formulas expand a $3 \times 3$ determinant in terms of $2 \times 2$ determinants, along a row of $A$. The attached signs $\pm 1$ are called the **checkerboard signs**, to be defined shortly. The $2 \times 2$ determinants are called **minors** of the $3 \times 3$ determinant $|A|$. The checkerboard sign together with a minor is called a **cofactor**.
Cofactor Expansion Illustration

Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the $2 \times 2$ determinants in the expansion. To illustrate, row 1 cofactor expansion gives

\[
\begin{vmatrix}
3 & 0 & 0 \\
2 & 1 & 7 \\
5 & 4 & 8
\end{vmatrix} = 3(+1) \begin{vmatrix}
1 & 7 \\
4 & 8
\end{vmatrix} + 0(-1) \begin{vmatrix}
2 & 7 \\
5 & 8
\end{vmatrix} + 0(+1) \begin{vmatrix}
2 & 1 \\
5 & 4
\end{vmatrix}
\]

\[
= 3(+1)(8 - 28) + 0 + 0
\]

\[
= -60.
\]

What has been said for rows also applies to columns, due to the transpose formula

\[
\det(A) = \det(A^T).
\]
Minor

The \((n - 1) \times (n - 1)\) determinant obtained from \(\text{det}(A)\) by striking out row \(i\) and column \(j\) is called the \((i, j)\)-minor of \(A\) and denoted \(\text{minor}(A, i, j)\). Literature might use \(M_{ij}\) for a minor.

Cofactor

The \((i, j)\)-cofactor of \(A\) is \(\text{cof}(A, i, j) = (-1)^{i+j} \text{minor}(A, i, j)\). Multiplicative factor \((-1)^{i+j}\) is called the checkerboard sign, because its value can be determined by counting \textit{plus}, \textit{minus}, \textit{plus}, etc., from location \((1, 1)\) to location \((i, j)\) in any checkerboard fashion.

Expansion of Determinants by Cofactors

\[
\text{det}(A) = \sum_{j=1}^{n} a_{kj} \text{cof}(A, k, j), \quad \text{det}(A) = \sum_{i=1}^{n} a_{i\ell} \text{cof}(A, i, \ell), \quad (11)
\]

In (11), \(1 \leq k \leq n\), \(1 \leq \ell \leq n\). The first expansion is called a \textit{cofactor row expansion} and the second is called a \textit{cofactor column expansion}. The value \(\text{cof}(A, i, j)\) is the cofactor of element \(a_{ij}\) in \(\text{det}(A)\), that is, the checkerboard sign times the minor of \(a_{ij}\).
2 Example (Hybrid Method) Justify by cofactor expansion and the four properties the identity

\[
\begin{vmatrix}
10 & 5 & 0 \\
11 & 5 & a \\
10 & 2 & b \\
\end{vmatrix}
= 5(6a - b).
\]

Solution: Let \( D \) denote the value of the determinant. Then

\[
D = \det \begin{pmatrix}
10 & 5 & 0 \\
11 & 5 & a \\
10 & 2 & b \\
\end{pmatrix}
\text{Given.}
\]

\[
= \det \begin{pmatrix}
10 & 5 & 0 \\
1 & 0 & a \\
0 & -3 & b \\
\end{pmatrix}
\text{Combination leaves the determinant unchanged: combo}(1,2,-1), \ combo(1,3,-1).
\]

\[
= \det \begin{pmatrix}
0 & 5 & -10a \\
1 & 0 & a \\
0 & -3 & b \\
\end{pmatrix}
\text{combo}(2,1,-10).
\]

\[
= (1)(-1) \det \begin{pmatrix}
5 & -10a \\
-3 & b \\
\end{pmatrix}
\text{Cofactor expansion on column 1.}
\]

\[
= (1)(-1)(5b - 30a)
\text{Sarrus’ rule for } n = 2.
\]

\[
= 5(6a - b).
\text{Formula verified.}
\]
3 Example (Cramer’s Rule) Solve by Cramer’s rule the system of equations

\[
\begin{align*}
2x_1 + 3x_2 + x_3 - x_4 &= 1, \\
x_1 + x_2 - x_4 &= -1, \\
3x_2 + x_3 + x_4 &= 3, \\
x_1 + x_3 - x_4 &= 0,
\end{align*}
\]

verifying \(x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 2\).
Solution: Form the four determinants $\Delta_1, \ldots, \Delta_4$ from the base determinant $\Delta$ as follows:

$$\Delta = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_1 = \det \begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta_2 = \det \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_3 = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \Delta_4 = \det \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Five repetitions of the methods used in the previous examples give the answers $\Delta = -2$, $\Delta_1 = -2$, $\Delta_2 = 0$, $\Delta_3 = -2$, $\Delta_4 = -4$, therefore Cramer’s rule implies the solution

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \quad x_4 = \frac{\Delta_4}{\Delta}.$$

Then $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$. 
Maple Code for Cramer’s Rule

The details of the computation above can be checked in computer algebra system Maple as follows.

```maple
with(linalg):
A := matrix(
    [[2, 3, 1, -1],
    [1, 1, 0, -1],
    [0, 3, 1, 1],
    [1, 0, 1, -1]]);
Delta := det(A);
b := vector([1, -1, 3, 0]);
B1 := A: col(B1, 1) := b;
Delta1 := det(B1);
x[1] := Delta1 / Delta;
```
The Adjugate Matrix

The adjugate \( \text{adj}(A) \) of an \( n \times n \) matrix \( A \) is the transpose of the matrix of cofactors,

\[
\text{adj}(A) = \begin{pmatrix}
\text{cof}(A, 1, 1) & \text{cof}(A, 1, 2) & \cdots & \text{cof}(A, 1, n) \\
\text{cof}(A, 2, 1) & \text{cof}(A, 2, 2) & \cdots & \text{cof}(A, 2, n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cof}(A, n, 1) & \text{cof}(A, n, 2) & \cdots & \text{cof}(A, n, n)
\end{pmatrix}^T.
\]

A cofactor \( \text{cof}(A, i, j) \) is the checkerboard sign \((-1)^{i+j}\) times the corresponding minor determinant \( \text{minor}(A, i, j) \).

Adjugate of a \( 2 \times 2 \)

\[
\text{adj} \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{pmatrix}
\]

In words: swap the diagonal elements and change the sign of the off–diagonal elements.
Adjugate Formula for the Inverse

For any \( n \times n \) matrix

\[
A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) I.
\]

The equation is valid even if \( A \) is not invertible. The relation suggests several ways to find \( \det(A) \) from \( A \) and \( \text{adj}(A) \) with one dot product.

For an invertible matrix \( A \), the relation implies \( A^{-1} = \frac{\text{adj}(A)}{\det(A)} \):

\[
A^{-1} = \frac{1}{\det(A)} \begin{pmatrix}
\text{cof}(A, 1, 1) & \text{cof}(A, 1, 2) & \cdots & \text{cof}(A, 1, n) \\
\text{cof}(A, 2, 1) & \text{cof}(A, 2, 2) & \cdots & \text{cof}(A, 2, n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cof}(A, n, 1) & \text{cof}(A, n, 2) & \cdots & \text{cof}(A, n, n)
\end{pmatrix}^T
\]
Application: Adjugate Shortcut

Given \( A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \), then we can compute \( \text{adj}(A) = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} \).

Suppose that we mark some unknown entries in \( \text{adj}(A) \) by \( ? \) and write \( |A| \) for \( \det(A) \). Then the formula \( A \text{adj}(A) = \text{adj}(A)A = \det(A)I \) becomes

\[
\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} = \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}.
\]

While the second product \( \text{adj}(A)A \) contains useless information, the first product gives \( \text{row}(A, 2) \text{col}(\text{adj}(A), 2) = \det(A) \). Because the values are known, then \( \det(A) = 6 + 1 + 0 = 7 \).

Knowing \( A \) and \( \text{adj}(A) \) gives the value of \( \det(A) \) in one dot product.
Elementary Matrices

Theorem 1 (Determinants and Elementary Matrices)
Let $E$ be an $n \times n$ elementary matrix. Then

- **Combination** $\det(E) = 1$
- **Multiply** $\det(E) = m$ for multiplier $m$.
- **Swap** $\det(E) = -1$
- **Product** $\det(EX) = \det(E) \det(X)$ for all $n \times n$ matrices $X$.

Theorem 2 (Determinants and Invertible Matrices)
Let $A$ be a given invertible matrix. Then

$$\det(A) = \frac{(-1)^s}{m_1m_2 \cdots m_r}$$

where $s$ is the number of swap rules applied and $m_1, m_2, \ldots, m_r$ are the nonzero multipliers used in multiply rules when $A$ is reduced to $rref(A)$. 
Theorem 3 (Determinant Product Rule)
Let $A$ and $B$ be given $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B).$$

Proof

Assume $A^{-1}$ does not exist. Then $A$ has zero determinant, which implies $\det(A) \det(B) = 0$. If $\det(B) = 0$, then $Bx = 0$ has infinitely many solutions, in particular a nonzero solution $x$. Multiply $Bx = 0$ by $A$, then $ABx = 0$ which implies $AB$ is not invertible. Then the identity $\det(AB) = \det(A) \det(B)$ holds, because both sides are zero. If $\det(B) \neq 0$ but $\det(A) = 0$, then there is a nonzero $y$ with $Ay = 0$. Define $x = B^{-1}y$. Then $ABx = Ay = 0$, with $x \neq 0$, which implies $AB$ is not invertible, and as earlier in this paragraph, the identity holds. This completes the proof when $A$ is not invertible.

Assume $A$ is invertible. In particular, $\text{rref}(A^{-1}) = I$. Write $I = \text{rref}(A^{-1}) = E_1E_2 \cdots E_kA^{-1}$ for elementary matrices $E_1, \ldots, E_k$. Then $A = E_1E_2 \cdots E_k$ and

(12) $$AB = E_1E_2 \cdots E_kB.$$  

The theorem follows from repeated application of the basic identity $\det(EX) = \det(E) \det(X)$ to relation (12), because

$$\det(AB) = \det(E_1) \cdots \det(E_k) \det(B) = \det(A) \det(B).$$