Solution Set Basis for Linear Differential Equations

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Linear Differential Equations

The solution set of a homogeneous constant coefficient linear differential equation

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0 \]

is known to be a vector space of functions of dimension \( n \), consisting of special linear combinations

\[ y = c_1 f_1 + \cdots + c_n f_n, \quad (1) \]

where \( f_1, \ldots, f_n \) are elementary functions known as atoms.
Definition of Atom

A base atom is defined to be one of

\[ 1, e^{ax}, \cos bx, \sin bx, e^{ax} \cos bx, e^{ax} \sin bx, \]

with real \( a \neq 0, b > 0 \).

An atom equals a base atom multiplied by \( x^n \), where \( n = 0, 1, 2 \ldots \) is an integer.

An atom has coefficient 1, and the zero function is not an atom.

Examples of Atoms

\[ 1, x, x^2, e^x, xe^{-x}, x^{15} e^{2x} \cos 3x, \cos 3x, \sin 2x, x^2 \cos 2x, x^6 \sin \pi x, x^{10} e^{\pi x} \sin 0.1x \]

Functions that are not Atoms

\[ x/(1 + x), \ln |x|, e^{x^2}, \sin(x + 1), 0, 2x, \sin(1/x), \sqrt{x} \]
Theorems about Atoms

Theorem 1 (Independence)
Any finite list of atoms is linearly independent.

Theorem 2 (Euler)
The real characteristic polynomial \( p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_0 \) has a factor \((r - a - ib)^{k+1}\) if and only if
\[
x^k e^{ax} \cos bx, \quad x^k e^{ax} \sin bx
\]
are real solutions of the differential equation (1). If \( b > 0 \), then both are atoms. If \( b = 0 \), then only the first is an atom.

Theorem 3 (Real Solutions)
If \( u \) and \( v \) are real and \( u + iv \) is a solution of equation (1), then \( u \) and \( v \) are real solutions of equation (1).

Theorem 4 (Basis)
The solution set of equation (1) has a basis of \( n \) solution atoms which are determined by Euler’s theorem.
Euler’s Theorem Translated

Theorem 5 (How to Apply Euler’s Theorem)

<table>
<thead>
<tr>
<th>Factor dividing $p(r)$</th>
<th>Solution Atom(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(r - 5)$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>$(r + 7)^2$</td>
<td>$e^{-7x}, xe^{-7x}$</td>
</tr>
<tr>
<td>$(r + 7)^3$</td>
<td>$e^{-7x}, xe^{-7x}, x^2e^{-7x}$</td>
</tr>
<tr>
<td>$r$</td>
<td>$e^{0x}$</td>
</tr>
<tr>
<td>$r^2$</td>
<td>$e^{0x}$ and $xe^{0x}$</td>
</tr>
<tr>
<td>$r^3$</td>
<td>$1, x$ and $x^2 [e^{0x} = 1]$</td>
</tr>
<tr>
<td>$(r - 5i)$</td>
<td>$\cos 5x$ and $\sin 5x$</td>
</tr>
<tr>
<td>$(r + 3i)^2$</td>
<td>$\cos 3x, x \cos 3x, \sin 3x, x \sin 3x$</td>
</tr>
<tr>
<td>$(r - 2 + 3i)^2$</td>
<td>$e^{2x} \cos 3x, xe^{2x} \cos 3x, e^{2x} \sin 3x, xe^{2x} \sin 3x$</td>
</tr>
</tbody>
</table>
Example 1. Solve $y''' = 0$. 

Solution: $p(r) = r^3$ implies 1 is a base atom and then $1, x, x^2$ are solution atoms. They are independent, hence form a basis for the 3-dimensional solution space. Then $y = c_1 + c_2x + c_3x^2$.

Example 2. Solve $y'' + 4y = 0$. 

Solution: $p(r) = r^2 + 4$ implies base atoms $\cos 2x$ and $\sin 2x$. They are a basis for the 2-dimensional solution space with $y = c_1 \cos 2x + c_2 \sin 2x$.

Example 3. Solve $y'' + 2y' = 0$. 

Solution: $p(r) = r^2 + 2r$ implies $1, e^{-2x}$ are base solution atoms. These independent atoms form a basis. Then $y = c_1 + c_2e^{-2x}$.

Example 4. Solve $y^{(4)} + 4y'' = 0$. 

Solution: $p(r) = r^4 + 4r^2 = r^2(r^2 + 4)$ implies the four atoms $1, x, \cos 2x, \sin 2x$ are solutions. Then $y = c_1 + c_2x + c_3 \cos 2x + c_3 \sin 2x$.

Example 5. Solve the differential equation if $p(r) = (r^3 - r^2)(r^2 - 1)(r^2 + 4)^2$. 

Solution: The distinct factors of $p(r)$ are $r^2, (r - 1)^2, r + 1, (r - 2i)^2, (r + 2i)^2$. Euler’s theorem implies the DE has nine solution atoms $1, x, e^x, xe^x, e^{-x}, \cos 2x, x \cos 2x, \sin 2x, x \sin 2x$. Then $y$ is a linear combination of these atoms.