Atoms

An **atom** is a term with coefficient 1 obtained by taking the real and imaginary parts of

$$x^j e^{ax+icx}, \quad j = 0, 1, 2, \ldots,$$

where a and c represent real numbers and $c \geq 0$.

Details and Remarks

- The definition plus Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies that an atom is a term of one of the following types:

  $$x^n, \; x^n e^{ax}, \; x^n e^{ax}\cos bx, \; x^n e^{ax}\sin bx.$$

  The symbol n is an integer $0, 1, 2, \ldots$ and a, b are real numbers with $b > 0$.

- In particular, $1, x, x^2, \ldots, x^k$ are atoms.

- The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom e^x. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms x and x^2 are indeed atoms. Terms like e^{x^2}, $\ln|x|$ and $x/(1 + x^2)$ are not atoms, nor are they constructed from atoms.

Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \quad \text{for all } x$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

Independence and Atoms

Theorem 1 (Atoms are Independent)
A list of finitely many distinct atoms is linearly independent.

Theorem 2 (Powers are Independent)
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent.

Theorem 3 (Homogeneous Solution y_h and Atoms)
Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of atoms.

Theorem 4 (Particular Solution y_p and Atoms)
A linear non-homogeneous differential equation with constant coefficients a having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 5 (General Solution y and Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{ a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{ a linear combination of atoms.}$$

Details

The first theorem follows from Picard's theorem, Euler's theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.
The second order **recipe** justifies the first theorem for the special case of second order differential equations, because $e^{r_1 x}$, $e^{r_2 x}$, $xe^{r_1 x}$, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are atoms.

How to Solve n-th Order Equations

- Picard's existence–uniqueness theorem says that $y''' + 2y'' + y = 0$ has general solution y constructed from $n = 3$ solutions of this differential equation. More precisely, the general solution of an n-th order linear differential equation is constructed from n solutions of the equation.
- Linear algebra says that the dimension of the solution set is this same fixed number n. Once n independent solutions are found for the differential equation, the search for the general solution has ended: y must be a linear combination of these n independent solutions.
- Because of the preceding structure theorems, we have reduced our search for the general solution as follows:

  Find n distinct atoms which are solutions of the differential equation.

Finding Solutions which are Atoms

Euler supplies us with a basic result, which tells us how to find the list of distinct atoms, which forms a basis of solutions of the linear differential equation.

Theorem 6 (Euler)
The function e^{rx} is a solution of a linear constant–coefficient differential equation if and only if r is a root of the characteristic equation.
More generally, the $k+1$ distinct atoms

$$e^{rx},\ xe^{rx},\ \ldots,\ x^k e^{rx}$$

are solutions if and only if r is a root of the characteristic equation of multiplicity $k+1$.

Theorem 7 (Complex Roots)
If $r = \alpha + i\beta$ is a complex root of multiplicity $k+1$, then the formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$e^{rx} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x).$$

Therefore, the $2k+2$ distinct atoms listed below are independent solutions of the differential equation:

$$\begin{array}{llll} e^{\alpha x}\cos(\beta x), & xe^{\alpha x}\cos(\beta x), & \ldots, & x^k e^{\alpha x}\cos(\beta x), \\ e^{\alpha x}\sin(\beta x), & xe^{\alpha x}\sin(\beta x), & \ldots, & x^k e^{\alpha x}\sin(\beta x) \end{array}$$