How to Solve Linear Differential Equations

• Definition: Base Atom, Atom
• Independence of Atoms
• Construction of the General Solution from a List of Distinct Atoms
• Euler’s Theorems
  – Euler’s Basic Theorem
  – Euler’s Multiplicity Theorem
  – A Shortcut Method
• Examples
• Main Theorems on Atoms and Linear Differential Equations
Atoms

A base atom is one of $1, e^{ax}, \cos bx, \sin bx, e^{ax} \cos bx, e^{ax} \sin bx$, with $b > 0$ and $a \neq 0$.

An atom equals $x^n$ times a base atom, where $n \geq 0$ is an integer.

Details and Remarks

- Euler’s formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies that an atom is constructed from the complex expression $x^n e^{ax+ibx}$ by taking real and imaginary parts.

- The powers $1, x, x^2, \ldots, x^k$ are atoms.

- The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom $e^x$. Zero is not an atom. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms $x$ and $x^2$ are indeed atoms. Terms like $-e^x, e^{-x^2}, x^{5/2} \cos x, \ln |x|$ and $x/(1 + x^2)$ are not atoms.
Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \text{ for all } x$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

Independence and Atoms

**Theorem 1 (Atoms are Independent)**

A list of finitely many distinct atoms is linearly independent.

**Theorem 2 (Powers are Independent)**

The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.
Construction of the General Solution from a List of Distinct Atoms

- **Picard’s theorem** says that the homogeneous constant-coefficient linear differential equation

\[ y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0 \]

has solution space \( S \) of dimension \( n \). Picard’s theorem reduces the general solution problem to finding \( n \) linearly independent solutions.

- **Euler’s theorem** *infra* says that the required \( n \) independent solutions can be taken to be atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [characteristic polynomial=left side]

\[ r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0 = 0. \]

- The **Fundamental Theorem of Algebra** states that there are exactly \( n \) roots \( r \), real or complex, for an \( n \)th order polynomial equation. The result implies that the characteristic equation has exactly \( n \) roots, counting multiplicities.

- **General Solution.** Because the list of atoms constructed by Euler’s theorem has \( n \) distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

\[ y = c_1(\text{atom 1}) + \cdots + c_n(\text{atom } n). \]

Symbols \( c_1, \ldots, c_n \) are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.
Euler’s Basic Theorem

Theorem 3 (L. Euler)
The exponential \( y = e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \( r = r_1 \) is a root of the characteristic equation.

- If \( r_1 = a \) is a real root, then one atom \( e^{ax} \) is constructed by Euler’s Theorem.
- If \( r_1 = a + ib \) is a complex root (\( b > 0 \)), then Euler’s Theorem gives a complex solution
  \[
e^{r_1 x} = e^{ax} \cos bx + ie^{ax} \sin bx.
  \]
  The real and imaginary parts of this complex solutions are real solutions of the differential equation. Therefore, one complex root \( r_1 = a + ib \) produces two atoms
  \[
e^{ax} \cos bx, \quad e^{ax} \sin bx.
  \]
  The conjugate root \( a - ib \) produces the same two atoms, hence it is ignored.
Euler’s Multiplicity Theorem

**Definition.** A root \( r = r_1 \) of a polynomial equation \( p(r) = 0 \) has multiplicity \( k \) provided \( (r - r_1)^k \) divides \( p(r) \) but \( (r - r_1)^{k+1} \) does not divide \( p(r) \). The calculus equivalent is \( p^{(j)}(r_1) = 0 \) for \( j = 0, \ldots, k - 1 \) and \( p^{(k)}(r_1) \neq 0 \).

**Theorem 4 (L. Euler)**
The expression \( y = x^k e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \( (r - r_1)^{k+1} \) divides the characteristic polynomial.
A Shortcut for using Euler’s Theorems

Let root $r_1$ of the characteristic equation have multiplicity $m + 1$.

- Find a base atom from Euler’s Basic Theorem.
- Multiply the base atom by $1, x, \ldots, x^m$.
  This process constructs, from the base atom, exactly $m + 1$ atoms. The atom count $m + 1$ equals the multiplicity of root $r_1$.
- A real root $r_1 = a$ will produce one base atom $e^{ax}$, to which this process is applied.
- A complex root $r_1 = a + ib$ will produce 2 base atoms $e^{ax} \cos bx, e^{ax} \sin bx$. This process is applied to both.
Atom List Examples

1. If root $r = -3$ has multiplicity 4, then the atom list is

$$e^{-3x}, xe^{-3x}, x^2 e^{-3x}, x^3 e^{-3x}.$$

The list is constructed by multiplying the base atom $e^{-3x}$ by powers $1, x, x^2, x^3$. The multiplicity 4 of the root equals the number of constructed atoms.

2. If $r = -3 + 2i$ is a root of the characteristic equation, then the base atoms for this root (both $-3 + 2i$ and $-3 - 2i$ counted) are

$$e^{-3x} \cos 2x, \quad e^{-3x} \sin 2x.$$

If root $r = -3 + 2i$ has multiplicity 3, then the two real atoms are multiplied by $1, x, x^2$ to obtain a total of 6 atoms

$$e^{-3x} \cos 2x, \quad xe^{-3x} \cos 2x, \quad x^2 e^{-3x} \cos 2x,$$
$$e^{-3x} \sin 2x, \quad xe^{-3x} \sin 2x, \quad x^2 e^{-3x} \sin 2x.$$

The number of atoms generated for each base atom is 3, which equals the multiplicity of the root $-3 + 2i$. 
Theorem 5 (Homogeneous Solution $y_h$ and Atoms)
Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of atoms.

Theorem 6 (Particular Solution $y_p$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients $a$ having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 7 (General Solution $y$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

\[ f(x) = \text{a linear combination of atoms} \]

has general solution

\[ y(x) = y_h(x) + y_p(x) = \text{a linear combination of atoms}. \]

Proofs
The first theorem follows from Picard’s theorem, Euler’s theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.