Fundamental Theorem of Calculus

Isaac Newton found these formulas in an effort to extend the gas mileage formula

$$D = RT, \qquad \textbf{Distance = Rate} \times \textbf{Time},$$

to instantaneous rates.

Definite Integrals	Indefinite Integrals
(a) $\int_a^b f'(x)dx = f(b) - f(a)$	$\int y'(x)dx = y(x) + C$
(b) $\left(\int_a^x g(t)dt\right)' = g(x)$	$\left(\int g(x)dx\right)' = g(x)$

Part (a) is used in differential equations $y' = f(x, y)$ to discover the solution $y(x)$. Part (b) is used in checking the answer.

Method of Quadrature

Also called the *integration method*, the idea is to multiply the differential equation by dx, then write an integral sign on each side. The logic is that equal integrands imply equal integrals.

- The method applies only to quadrature equations $y' = F(x)$.

  Quadrature Classification Test
  The equation $y' = f(x, y)$ is a quadrature equation if and only if function $f(x, y)$ is independent of y, or equivalently, $\frac{\partial f}{\partial y} = 0$.

- The Fundamental Theorem of Calculus is applied on the left side to evaluate $\int y'(x)dx = y(x) + C$, where C is a constant.
- The method finds a candidate solution $y(x)$. It does not verify that the expression works.

Example: Method of Quadrature

1 Example (Quadrature) Solve $y' = 2x$ by the method of quadrature.

- Multiply $y' = 2x$ by dx, then write an integral sign on each side.

$$\int y'(x)dx = \int 2xdx$$

- Apply the FTC $\int y'(x)dx = y(x) + C$ on the left:

$$y(x) + c_1 = \int 2xdx$$

- Evaluate the integral on the right by tables. Then

$$y(x) + c_1 = x^2 + c_2, \quad \text{or} \quad y(x) = x^2 + C$$

2 Example (Quadrature) Solve $y' = 3e^x$, $y(0) = 2$.

Candidate solution. The *method of quadrature* is applied.

$y'(x) = 3e^x$	Copy the equation.
$y'(x)dx = 3e^x dx$	Multiply both sides by dx.
$\int y'(x)dx = \int 3e^x dx$	Add an integral on each side.
$y(x) + c_1 = 3e^x + c_2$	Fundamental theorem of calculus (FTC) used left. Integral table used right.
$y(x) = 3e^x + C$	Quadrature complete. Next, find C.
$2 = y(0) = 3e^0 + C$	Substitute $x = 0$. Use $y(0) = 2$.
$y(x) = 3e^x - 1$	Substitute $C = -1$. Solution candidate found.

Candidate Solution: $\boxed{y(x) = 3e^x - 1}$

Two-Panel Answer Check

A typical answer check involves two panels, because two equations must be tested: (1) The differential equation, and (2) The initial condition. Abbreviations LHS=*Left-Hand-side* and RHS=*Right-Hand-Side* are used in the displays.

Verify DE. Panel 1 of the answer check tests the solution $y = 3e^x - 1$ of the differential equation (DE) $y' = 3e^x$:

$\text{LHS} = y'$	Left side of the differential equation.
$= (3e^x - 1)'$	Substitute $y = 3e^x - 1$.
$= 3e^x - 0$	Sum rule, constant rule and $(e^u)' = u'e^u$.
$= \text{RHS}$	DE verified.

Verify IC. Panel 2 of the answer check tests the initial condition (IC) $y(0) = 2$:

$\text{LHS} = y(0)$	Left side of the initial condition $y(0) = 2$.
$= (3e^x - 1)\vert_{x=0}$	Substitute $y = 3e^x - 1$.
$= 3e^0 - 1$	
$= 2$	Simplify using $e^0 = 1$.
$= \text{RHS}$	IC verified.