

Stability of Dynamical systems

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Stability

Consider an autonomous system $\vec{u}'(t) = \vec{f}(\vec{u}(t))$ with \vec{f} continuously differentiable in a region D in the plane.

Stable equilibrium. An equilibrium point \vec{u}_0 in D is said to be **stable** provided for each $\epsilon > 0$ there corresponds $\delta > 0$ such that (a) and (b) hold:

- (a) Given $\vec{u}(0)$ in D with $\|\vec{u}(0) - \vec{u}_0\| < \delta$, then $\vec{u}(t)$ exists on $0 \leq t < \infty$.
- (b) Inequality $\|\vec{u}(t) - \vec{u}_0\| < \epsilon$ holds for $0 \leq t < \infty$.

Unstable equilibrium. The equilibrium point \vec{u}_0 is called **unstable** provided it is **not stable**, which means (a) or (b) fails (or both).

Asymptotically stable equilibrium. The equilibrium point \vec{u}_0 is said to be **asymptotically stable** provided (a) and (b) hold (it is **stable**), and additionally

- (c) $\lim_{t \rightarrow \infty} \|\vec{u}(t) - \vec{u}_0\| = 0$ for $\|\vec{u}(0) - \vec{u}_0\| < \delta$.

Isolated equilibria

An autonomous system is said to have an **isolated equilibrium** at $\vec{u} = \vec{u}_0$ provided \vec{u}_0 is the only constant solution of the system in $|\vec{u} - \vec{u}_0| < r$, for $r > 0$ sufficiently small.

Theorem 1 (Isolated Equilibrium)

The following are equivalent for a constant planar system $\vec{u}'(t) = A\vec{u}(t)$:

1. The system has an isolated equilibrium at $\vec{u} = \vec{0}$.
2. $\det(A) \neq 0$.
3. The roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ satisfy $\lambda_1 \lambda_2 \neq 0$.

Proof: The expansion $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2$ shows that $\det(A) = \lambda_1 \lambda_2$. Hence **2** \equiv **3**. We prove now **1** \equiv **2**. If $\det(A) = 0$, then $A\vec{u} = \vec{0}$ has infinitely many solutions \vec{u} on a line through $\vec{0}$, therefore $\vec{u} = \vec{0}$ is not an isolated equilibrium. If $\det(A) \neq 0$, then $A\vec{u} = \vec{0}$ has exactly one solution $\vec{u} = \vec{0}$, so the system has an isolated equilibrium at $\vec{u} = \vec{0}$.

Classification of Isolated Equilibria

For linear equations

$$\vec{u}'(t) = A\vec{u}(t),$$

we explain the phase portrait classifications

saddle, node, spiral, center

near an isolated equilibrium point $\vec{u} = \vec{0}$, and how to detect these classifications, when they occur.

Symbols λ_1, λ_2 are the roots of $\det(A - \lambda I) = 0$.

Atoms corresponding to roots λ_1, λ_2 happen to classify the phase portrait as well as its stability. A **shortcut** will be explained to determine a classification, *based only on the atoms*.

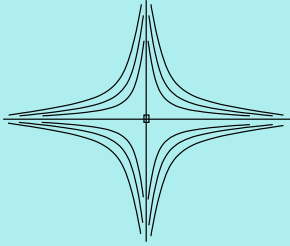


Figure 1. Saddle

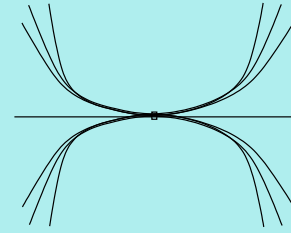


Figure 2. Improper node

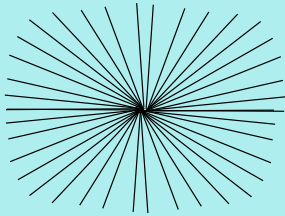


Figure 3. Proper node

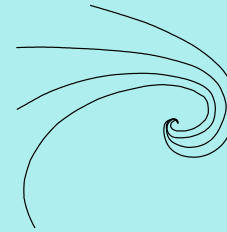


Figure 4. Spiral

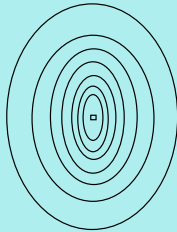


Figure 5. Center

Saddle λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$

A **saddle** has solution formula

$$\vec{u}(t) = e^{\lambda_1 t} \vec{c}_1 + e^{\lambda_2 t} \vec{c}_2,$$

$$\vec{c}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} \vec{u}(0), \quad \vec{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} \vec{u}(0).$$

The phase portrait shows two lines through the origin which are tangents at $t = \pm\infty$ for all orbits.

A saddle is **unstable** at $t = \infty$ and $t = -\infty$, due to the limits of the atoms $e^{r_1 t}, e^{r_2 t}$ at $t = \pm\infty$.

Node λ_1, λ_2 real, $\lambda_1 \lambda_2 > 0$ **Case** $\lambda_1 = \lambda_2$. An **improper node** has solution formula

$$\vec{u}(t) = e^{\lambda_1 t} \vec{c}_1 + t e^{\lambda_1 t} \vec{c}_2,$$

$$\vec{c}_1 = \vec{u}(0), \quad \vec{c}_2 = (A - \lambda_1 I) \vec{u}(0).$$

An improper node is further classified as a **degenerate node** ($\vec{c}_2 \neq \vec{0}$) or a **star node** ($\vec{c}_2 = \vec{0}$). Discussed below is **subcase** $\lambda_1 = \lambda_2 < 0$. For **subcase** $\lambda_1 = \lambda_2 > 0$, replace ∞ by $-\infty$.

degenerate node

A phase portrait has all trajectories tangent at $t = \infty$ to direction \vec{c}_2 .

star node

A phase portrait consists of trajectories $\vec{u}(t) = e^{\lambda_1 t} \vec{c}_1$, a straight line, with limit $\vec{0}$ at $t = \infty$. Vector \vec{c}_1 can be any direction.

Node

λ_1, λ_2 real, $\lambda_1 \lambda_2 > 0$

Case $\lambda_1 \neq \lambda_2$. A **proper node** is any node that is not improper. Its solution formula is

$$\vec{u}(t) = e^{\lambda_1 t} \vec{c}_1 + e^{\lambda_2 t} \vec{c}_2,$$

$$\vec{c}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} \vec{u}(0), \quad \vec{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} \vec{u}(0).$$

- A trajectory near a proper node satisfies, for some direction \vec{v} , $\lim_{t \rightarrow \omega} \vec{u}'(t) / |\vec{u}'(t)| = \vec{v}$, for either $\omega = \infty$ or $\omega = -\infty$. Briefly, $\vec{u}(t)$ is tangent to \vec{v} at $t = \omega$.
- To each direction \vec{v} corresponds some $\vec{u}(t)$ tangent to \vec{v} .

Spiral $\lambda_1 = \bar{\lambda}_2 = a + ib$ complex, $a \neq 0$, $b > 0$.

A **spiral** has solution formula

$$\vec{u}(t) = e^{at} \cos(bt) \vec{c}_1 + e^{at} \sin(bt) \vec{c}_2,$$

$$\vec{c}_1 = \vec{u}(0), \quad \vec{c}_2 = \frac{A - aI}{b} \vec{u}(0).$$

All solutions are bounded harmonic oscillations of natural frequency b times an exponential amplitude which grows if $a > 0$ and decays if $a < 0$. An orbit in the phase plane **spirals out** if $a > 0$ and **spirals in** if $a < 0$.

Center $\lambda_1 = \bar{\lambda}_2 = a + ib$ complex, $a = 0, b > 0$

A **center** has solution formula

$$\vec{u}(t) = \cos(bt) \vec{c}_1 + \sin(bt) \vec{c}_2,$$

$$\vec{c}_1 = \vec{u}(0), \quad \vec{c}_2 = \frac{1}{b} A\vec{u}(0).$$

All solutions are bounded harmonic oscillations of natural frequency b . Orbits in the phase plane are periodic closed curves of period $2\pi/b$ which encircle the origin.

Attractor and Repeller

- An equilibrium point is called an **attractor** provided solutions starting nearby limit to the point as $t \rightarrow \infty$.
- A **repeller** is an equilibrium point such that solutions starting nearby limit to the point as $t \rightarrow -\infty$.
- Terms like **attracting node** and **repelling spiral** are defined analogously.

Almost linear systems

A nonlinear planar autonomous system $\vec{u}'(t) = \vec{f}(\vec{u}(t))$ is called **almost linear** at equilibrium point $\vec{u} = \vec{u}_0$ if there is a 2×2 matrix A and a vector function \vec{g} such that

$$\vec{f}(\vec{u}) = A(\vec{u} - \vec{u}_0) + \vec{g}(\vec{u}),$$
$$\lim_{\|\vec{u} - \vec{u}_0\| \rightarrow 0} \frac{\|\vec{g}(\vec{u})\|}{\|\vec{u} - \vec{u}_0\|} = 0.$$

The function \vec{g} has the same smoothness as \vec{f} .

We investigate the possibility that a local phase diagram at $\vec{u} = \vec{u}_0$ for the nonlinear system $\vec{u}'(t) = \vec{f}(\vec{u}(t))$ is graphically identical to the one for the linear system $\vec{y}'(t) = A\vec{y}(t)$ at $\vec{y} = \vec{0}$.

Jacobian Matrix

Almost linear system results will apply to **all isolated equilibria** of $\vec{u}'(t) = \vec{f}(\vec{u}(t))$. This is accomplished by expanding \vec{f} in a Taylor series about each equilibrium point, which implies that the ideas are applicable to different choices of \mathbf{A} and \mathbf{g} , depending upon which equilibrium point \vec{u}_0 was considered.

Define the **Jacobian matrix** of \vec{f} at equilibrium point \vec{u}_0 by the formula

$$\mathbf{J} = \text{aug} \left(\partial_1 \vec{f}(\vec{u}_0), \partial_2 \vec{f}(\vec{u}_0) \right).$$

Taylor's theorem for functions of two variables says that

$$\vec{f}(\vec{u}) = \mathbf{J}(\vec{u} - \vec{u}_0) + \vec{g}(\vec{u})$$

where $\vec{g}(\vec{u})/\|\vec{u} - \vec{u}_0\| \rightarrow \mathbf{0}$ as $\|\vec{u} - \vec{u}_0\| \rightarrow \mathbf{0}$. Therefore, for \vec{f} continuously differentiable, we may always take $\mathbf{A} = \mathbf{J}$ to obtain from the almost linear system $\vec{u}'(t) = \vec{f}(\vec{u}(t))$ its **linearization** $\mathbf{y}'(t) = \mathbf{A}\vec{y}(t)$.