Introduction to Dynamical Systems

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Autonomous Planar Systems

A set of two scalar differential equations of the form

$$\begin{aligned} x'(t) &= F(x(t), y(t)), \\ y'(t) &= G(x(t), y(t)). \end{aligned} \tag{1}$$

is called a **planar autonomous system**.

The term **autonomous** means **self-governing**, justified by the absence of the time variable t in the functions $F(x, y)$, $G(x, y)$.

Vector form of a Dynamical System

To obtain the vector form, let

$$\vec{\mathrm{u}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \vec{\mathrm{f}}(x,y) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}.$$

Then write system

$$x'(t) = F(x(t), y(t)), \quad y'(t) = G(x(t), y(t))$$

as the first order vector-matrix system

(2) $$\vec{\mathrm{u}}'(t) = \vec{\mathrm{f}}(\vec{\mathrm{u}}(t)).$$

It is assumed that F, G are continuously differentiable in some region D in the xy-plane.

Trajectories

The assumption that F and G are continuously differentiable implies $\vec{\mathbf{f}}(\vec{\mathbf{u}})$ is continuously differentiable in D and guarantees that Picard's existence-uniqueness theorem for initial value problems applies to the initial value problem

$$\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t)), \quad \vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0.$$

Accordingly, to each $\vec{\mathbf{u}}_0 = (x_0, y_0)$ in D there corresponds a unique solution $\vec{\mathbf{u}}(t) = (x(t), y(t))$, represented as a planar curve in the xy-plane, which passes through planar position $\vec{\mathbf{u}}_0 = (x_0, y_0)$ at time $t = 0$. These curves are called **solution trajectories**.

Trajectories Don't Cross

Hand-drawn phase portraits are limited: *you cannot draw a solution trajectory that touches another solution curve*!

Theorem 1 (Identical Trajectories)
Assume that Picard's existence-uniqueness theorem applies to initial value problems in D for the planar system

$$\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t)), \quad \vec{\mathbf{u}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two trajectories of the system. If times t_1, t_2 exist such that

$$x_1(t_1) = x_2(t_2), \quad y_1(t_1) = y_2(t_2), \tag{3}$$

then for the value $c = t_1 - t_2$ the equations $x_1(t+c) = x_2(t)$ and $y_1(t+c) = y_2(t)$ are valid for all allowed values of t. This means that the two trajectories are on one and the same planar curve, or in the contrapositive, two different trajectories cannot touch or cross in the phase plane.

Proof

Define $x(t) = x_1(t + c)$, $y(t) = y_1(t + c)$. By the chain rule, $(x(t), y(t))$ is a solution of the planar system, because

$$x'(t) = x_1'(t + c) = F(x_1(t + c), y_1(t + c)) = F(x(t), y(t)),$$

and similarly for the second differential equation. Assumed equations

$$x_1(t_1) = x_2(t_2), \quad y_1(t_1) = y_2(t_2)$$

imply

$$x(t_2) = x_2(t_2), \quad y(t_2) = y_2(t_2),$$

therefore Picard's uniqueness theorem implies that $x(t) = x_2(t)$ and $y(t) = y_2(t)$ for all allowed values of t. The proof is complete.

Equilibria

A trajectory that reduces to a point, or a constant solution $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}_0$, $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{y}_0$, is called an **equilibrium solution**. The equilibrium solutions or **equilibria** are found by solving the nonlinear equations

$$\boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{y}_0) = 0, \quad \boldsymbol{G}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \mathbf{0}.$$

Each such $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ in $\boldsymbol{D}$ is a trajectory whose graphic in the phase plane is a single point, called an **equilibrium point**. In applied literature, it may be called a **critical point**, **stationary point** or **rest point**. Theorem 1 has the following geometrical interpretation.

Assuming uniqueness, no other trajectory $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t}))$ in the phase plane can touch an equilibrium point $(\boldsymbol{x}_0, \boldsymbol{y}_0)$.

Population Biology

Planar autonomous systems have been applied to two-species populations like two species of trout, who compete for food from the same supply, and foxes and rabbits, who compete in a predator-prey situation.

Certain equilibria are significant, because they represent the population sizes for **cohabitation**. A point in the phase space that is not an equilibrium point corresponds to population sizes that cannot coexist, they must change with time. Some equilibria are consequently **observable** or **average** population sizes while non-equilibria correspond to snapshot population sizes that are subject to flux.

Biologists expect population sizes of such two-species competition models to undergo change until they reach approximately the observable values.

Rabbit-Fox System

Typical predator-prey equations for x rabbits and y foxes are given by the system of differential equations

$$(4) \qquad \begin{aligned} x'(t) &= 0.004\,(40 - y(t))x(t), \\ y'(t) &= 0.02\,(x(t) - 60)y(t). \end{aligned}$$

This example is called a **predator-prey** system, in which the expected observable population sizes $x = 60$, $y = 40$ are averages, about which the actual populations size oscillate, periodically over time. The equilibrium $x = 60$, $y = 40$ represents **ideal cohabitation**.

- Biological experiments suggest that initial population sizes close to the equilibrium values cause populations to stay near the initial sizes, even though the populations oscillate periodically.
- Observations by biologists of large population variations seem to verify that individual populations oscillate periodically around the ideal cohabitation sizes.

Trout System

Consider a population of two species of trout who compete for the same food supply. A typical autonomous planar system for the species x and y is

$$\begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

Equilibria. The equilibrium solutions for this system are

$$(0, 0), \quad (90, 0), \quad (0, 60), \quad (80, 20).$$

Only nonnegative population sizes are physically significant. Units for the population sizes might be in hundreds or thousands of fish. The equilibrium $(0, 0)$ corresponds to **extinction** of both species, while $(0, 60)$ and $(90, 0)$ correspond to the unusual situation of extinction for one species. The last equilibrium $(80, 20)$ corresponds to **co-existence** of the two trout species with observable population sizes of 80 and 20.

Trout System Phase Portrait

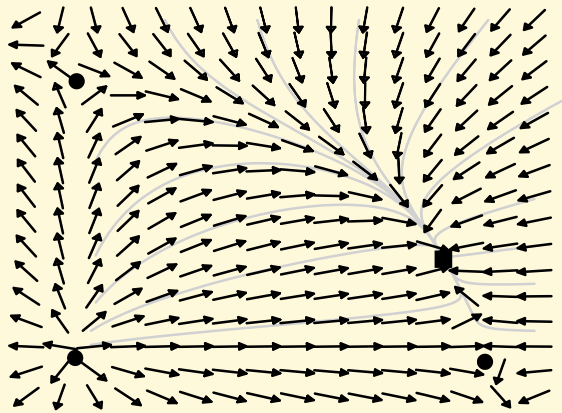

Figure 1. Phase portrait for the trout system.

Shown are typical solution curves and a direction field. Equilibrium $(80, 20)$ (a square) is observable, representing average population sizes. Equilibria $(0, 0)$, $(90, 0)$, $(0, 60)$ (circles) are not observable, representing unusual instances of population sizes.