The Undetermined Coefficients
Trial Solution Method

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**Definition of Solution Atom**

A solution atom of a linear constant-coefficient homogeneous differential equation is briefly called an **atom**. The set of atoms is generated from base atoms and powers of $x$.

A **base atom** is one of the terms $1, \cos bx, \sin bx, e^{ax}, e^{ax} \cos bx, e^{ax} \sin bx$.

An **atom** equals $x^n$ times a base atom, for $n = 0, 1, 2, 3 \ldots$.

**Examples.**

The following are atoms: $e^{2x}, e^{e^2x}, xe^{-\pi x}, e^{0x}$ or $1, x, x^2, \cos x, \cos \pi x, e^{-x} \sin 2x, x^6 \sin 100x, x^2e^{-5x}, x^5e^{-5x} \cos 5x, 2^x$ [equals $e^{ax}$ with $a = \ln 2$], any power $x^n$ with integer $n \geq 0$.

The following are not atoms: $2, x^{-1}, \ln |x|, e^{x^2}, \tan x, \sinh x, \sec x, \csc x, \sin^2 x, \sin(x^2), e^x \cos(2x + 2), \cot x, \frac{x}{1+x}$. 
Step 1. Find a trial solution $y$ by Rule I.

Rule I. Assume the right side $f(x)$ of the differential equation is a linear combination of atoms. Make a list of all distinct atoms that appear in the derivatives $f(x), f'(x), f''(x), \ldots$. Multiply these $k$ atoms by undetermined coefficients $d_1, \ldots, d_k$, then add to define a trial solution $y$.

Warning: Rule I can Fail. It fails exactly when one of the atoms is a solution of the homogeneous differential equation. Do Rule II infra first, in case of failure of Rule I.

Step 2. Substitute trial solution $y$ into the differential equation. The resulting equation is a competition between two linear combinations of the $k$ atoms in the list.

Step 3. Linear independence of atoms implies matching of coefficients of atoms left and right. Write out linear algebraic equations for unknowns $d_1, d_2, \ldots, d_k$. Solve the equations.

Step 4. The trial solution $y$ with evaluated coefficients $d_1, d_2, \ldots, d_k$ becomes the particular solution $y_p$. 
Rule I Failure

Example. The differential equation $y'' = x + e^x$ has by Rule I a trial solution $y = d_1(1) + d_2(x) + d_3(e^x)$ obtained from the list of $k = 3$ atoms $1, x, e^x$. The trial solution fails to work, because upon substitution of $y$ into the differential equation the resulting equation is

$$d_1(1)'' + d_2(x)'' + d_3(e^x)'' = 0(1) + 1(x) + 1(e^x).$$

This equation cannot be satisfied by choosing values of $d_1, d_2, d_3$, because it reads

$$x + (1 - d_3)e^x = 0,$$

implying that $x, e^x$ are dependent, a violation of the Independence of Atoms Theorem.

The actual trouble is a deeper problem. The equations $(1)'' = 0$ and $(x)'' = 0$ imply that $1$ and $x$ are solutions of the homogeneous differential equation $y'' = 0$. These equations cause constants $d_1, d_2$ to be completely absent from the system of equations. The constants $d_1, d_2, d_3$ must be uniquely determined. A variable that is absent in a linear system is a free variable, causing non-uniqueness, and this is the root of the problem.
Symbols

The symbols $c_1$, $c_2$ are reserved for use as arbitrary constants in the general solution $y_h$ of the homogeneous equation. For example, the homogeneous equation $y'' + y = 0$ has general solution $y = c_1 \cos x + c_2 \sin x$.

Symbols $d_1, d_2, d_3, \ldots$ are reserved for use in the trial solution $y$ of the non-homogeneous equation. For example, the equation $y'' + y = x + e^x$ has by Rule I trial solution $y = d_1(1) + d_2(x) + d_3(e^x)$.

Abbreviations: $c =$ constant, $d =$ determined.
Superposition

The relation \( y = y_h + y_p \) suggests solving \( a y'' + b y' + c y = f(x) \) in two stages:

(a) Find \( y_h \) as a linear combination of atoms computed by applying Euler’s theorem to factors of the characteristic polynomial \( ar^2 + br + c \).

(b) Apply the **undetermined coefficients trial solution method** to find \( y_p \).

- We expect to find two arbitrary constants \( c_1, c_2 \) in the solution \( y_h \), but in contrast, **no arbitrary constants** appear in \( y_p \).
- Calling \( d_1, d_2, d_3, \ldots \) undetermined coefficients is misleading, because in fact they are eventually determined.
The Trial Solution with Fewest Atoms

Undetermined coefficient theory computes a **shortest possible trial solution**, a solution with **fewest atoms**.

Using the fewest atoms minimizes the size of the linear algebra problem for the constants \(d_1, \ldots, d_k\).

**Example.** \(y'' + y = x^2\)

The atom list for \(f(x) = x^2\) is \(1, x, x^2\). Rule I computes a shortest trial solution \(y = d_1 + d_2 x + d_3 x^2\). The linear algebra problem is \(3 \times 3\), and no smaller system of equations can be found.
The Rules for Undetermined Coefficients

**Rule I.** Assume the right side $f(x)$ of the differential equation is a linear combination of atoms. Make a list of all distinct atoms that appear in the derivatives $f(x), f'(x), f''(x), \ldots$. Multiply these $k$ atoms by undetermined coefficients $d_1, \ldots, d_k$, then add to define a trial solution $y$.

This rule **FAILS** if one or more of the $k$ atoms is a solution of the homogeneous differential equation.

**Rule II.** If Rule I **FAILS**, then break the $k$ atoms into groups with the same **base atom**. Cycle through the groups, replacing atoms as follows. If the first atom in the group is a solution of the homogeneous differential equation, then multiply all atoms in the group by factor $x$. Repeat until the first atom is not a solution of the homogeneous differential equation. Multiply the constructed $k$ atoms by symbols $d_1, \ldots, d_k$ and add to define trial solution $y$. 
Assume the constant-coefficient differential equation has order 2 and the trial solution from Rule I contains 4 groups of 7 atoms

<table>
<thead>
<tr>
<th>Group</th>
<th>Atoms</th>
<th>Base Atom</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e^{2x}$, $xe^{2x}$, $x^2e^{2x}$, $x^3e^{ex}$</td>
<td>$e^{2x}$</td>
</tr>
<tr>
<td>2</td>
<td>$\cos x$</td>
<td>$\cos x$</td>
</tr>
<tr>
<td>3</td>
<td>$\sin x$</td>
<td>$\sin x$</td>
</tr>
<tr>
<td>4</td>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
</tbody>
</table>
Example 1

Assume the characteristic equation of the second order homogeneous differential equation is

\[(r - 1)(r - 3) = 0.\]

Then \(e^{2x}\), \(\cos x\), \(\sin x\) are not solutions of the homogeneous equation, but \(e^x\) and \(e^{3x}\) are solutions.

**Rule I fails** because the Group 4 atom \(e^x\) is a solution of the homogeneous equation. The other groups do not contain solutions of the homogeneous differential equation.

**Rule II applies** to give one new group and three unchanged groups. The trial solution \(y\) is a linear combination of the 7 atoms.

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(e^{2x}), (xe^{2x}), (x^2e^{2x}), (x^3e^{ex})</td>
</tr>
<tr>
<td>2</td>
<td>(\cos x)</td>
</tr>
<tr>
<td>3</td>
<td>(\sin x)</td>
</tr>
<tr>
<td>New 4</td>
<td>(xe^x)</td>
</tr>
</tbody>
</table>

**Details.** Atom \(xe^x\) is a solution of the homogeneous equation if and only if 1 is a double root of the characteristic equation; it isn’t, which stops the multiplication by \(x\) in Group 4.
Example 2

Assume the characteristic equation of the second order homogeneous differential equation is

$$(r - 1)(r - 2) = 0.$$  

Then $e^x$, $e^{2x}$ are solutions of the homogeneous equation, but $\cos x$, $\sin x$ are not solutions.

**Rule I fails** because the Group 1 atom $e^{2x}$ is a solution of the homogeneous equation. **Rule II applies** to give two new groups and two unchanged groups. The trial solution $y$ is a linear combination of the 7 atoms.

<table>
<thead>
<tr>
<th>Group</th>
<th>Atoms</th>
</tr>
</thead>
<tbody>
<tr>
<td>New 1</td>
<td>$xe^{2x}$, $x^2e^{2x}$, $x^3e^{2x}$, $x^4e^{ex}$</td>
</tr>
<tr>
<td>2</td>
<td>$\cos x$</td>
</tr>
<tr>
<td>3</td>
<td>$\sin x$</td>
</tr>
<tr>
<td>New 4</td>
<td>$xe^x$</td>
</tr>
</tbody>
</table>

**Details.** Atom $xe^{2x}$ is a solution of the homogeneous equation if and only if 2 is a double root of the characteristic equation; it isn’t, which stops the multiplication by $x$ in Group 1.

Atom $xe^x$ is a solution of the homogeneous equation if and only if 1 is a double root of the characteristic equation; it isn’t, which stops the multiplication by $x$ in Group 4.