The Corrected Trial Solution in the Method of Undetermined Coefficients

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Definition of Related Atoms

A base atom is one of the terms $1, \cos bx, \sin bx, e^{ax}, e^{ax} \cos bx, e^{ax} \sin bx$. An atom equals $x^n$ times a base atom, for $n = 0, 1, 2, 3, \ldots$.

Atoms $A$ and $B$ are related if and only if their successive derivatives $A, A', A'', \ldots, B, B', B'', \ldots$ share a common atom.

Then $x^3$ is related to $x$ and $x^{101}$, while $x$ is unrelated to $e^x, xe^x$ and $x \sin x$. Atoms $x \sin x$ and $x^3 \cos x$ are related, but the atoms $\cos 2x$ and $\sin x$ are unrelated.

An easy way to detect related atoms:

Atom $A$ is related to atom $B$ if and only if their base atoms are identical or else they would become identical by changing a sine to a cosine.
The Basic Trial Solution Method

The method is outlined here for an \( n \)th order linear differential equation.

**Undetermined Coefficients Trial Solution Method**

**Step 1.** Let \( g(x) = x^n f(x) \), where \( n \) is the order of the differential equation.

Extract all distinct atoms that appear in the derivatives \( g(x), g'(x), g''(x), \ldots \), then collect the distinct atoms so found into a list of \( k \) atoms. Multiply these atoms by **undetermined coefficients** \( d_1, \ldots, d_k \), then add to define a **trial solution** \( y \).

**Step 2.** Substitute \( y \) into the differential equation.

**Step 3.** Match coefficients of atoms left and right to write out linear algebraic equations for unknowns \( d_1, d_2, \ldots, d_k \). Solve the equations. Any variables not appearing are set to zero.

**Step 4.** The trial solution \( y \) with evaluated coefficients \( d_1, d_2, \ldots, d_k \) becomes the particular solution \( y_p \).
Symbols

The symbols $c_1, c_2$ are reserved for use as arbitrary constants in the general solution $y_h$ of the homogeneous equation.

Symbols $d_1, d_2, d_3, \ldots$ are reserved for use in the trial solution $y$ of the non-homogeneous equation. Abbreviations: $c =$ constant, $d =$ determined.
Superposition

The relation \( y = y_h + y_p \) suggests solving \( ay'' + by' + cy = f(x) \) in two stages:

(a) Find \( y_h \) as a linear combination of atoms computed by applying Euler’s theorem to factors of the characteristic polynomial \( ar^2 + br + c \).

(b) Apply the **basic trial solution method** to find \( y_p \).

- We expect to find two arbitrary constants \( c_1, c_2 \) in the solution \( y_h \), but in contrast, no arbitrary constants appear in \( y_p \).
- Calling \( d_1, d_2, d_3, \ldots \) undetermined coefficients is misleading, because in fact they are eventually determined.
The Trial Solution with Fewest Atoms

Undetermined coefficient theory computes a trial solution with fewest atoms, thereby eliminating superfluous symbols, which effects a reduction in the size of the algebra problem. In the case of the example $y'' + y = x^2$, the theory computes a trial solution $y = d_1 + d_2x + d_3x^2$, reducing the number of symbols from 5 to 3.

In a general equation $ay'' + by' + cy = f(x)$, the atoms in the trial solution $y$ are the atoms that appear in $g(x) = x^2f(x)$ plus all lower-power related atoms. Equivalently, the atoms are those extracted from the successive derivatives $g(x)$, $g'(x)$, $g''(x)$, …. For example, if $f(x) = x^2$, then $g(x) = x^2(x^2) = x^4$ and the list of derivatives is $x^4$, $4x^3$, $12x^2$, $24x$, $24$. Strip coefficients to identify the list of related atoms 1, $x$, $x^2$, $x^3$, $x^4$. Alternatively, begin with the atoms in $g(x)$, namely $x^4$, and append all lower powered related atoms. Briefly, atom $x^4$ causes an append of related atoms 1, $x$, $x^2$, $x^3$. 
Two Correction Rules

The initial trial solution $y$ obtained by constructing atoms from $g(x) = x^n f(x)$ is not the trial solution with fewest atoms. It is a sum of terms which can be organized into groups of related atoms, and it is known that each group contains $n$ superfluous terms. The correction rules describe how to remove the superfluous terms, which produces the desired corrected trial solution with fewest possible atoms.
Correction Rule I

If some variable $d_p$ is missing after substitution Step 2, then the system of linear equations for $d_1, \ldots, d_k$ fails to have a unique solution. In the language of linear algebra, a missing variable $d_p$ in the system of linear equations is a free variable, which implies the linear system in the unknowns $d_1, \ldots, d_k$ has, among the three possibilities, infinitely many solutions.

A symbol $d_p$ appearing in a trial solution will be missing in Step 2 if and only if it multiplies an atom $A(x)$ that is a solution of the homogeneous equation. Because $d_p$ will be a free variable [any missing variable is a free variable], to which we will assign value zero in Step 3, the term $d_p A(x)$ can be removed from the trial solution. We can do this in advance, to decrease the number of symbols in the trial solution.

Rule I. Remove all terms $d_p A(x)$ in the trial solution of Step 1 for which atom $A(x)$ is a solution of the homogeneous differential equation.
Correction Rule II

The trial solution always contains superfluous atoms, introduced by using $x^n f(x)$ to construct the trial solution instead of $f(x)$. For example, the equation $y'' + y = x^2$ would have trial solution $y = d_1 + d_2 x + d_3 x^2 + d_4 x^3 + d_5 x^4$, with atoms $x^3$ and $x^4$ superfluous, because $y_p = x^2 - 2$. We could have replaced the 5-term trial solution by 3-termed trial solution $y = d_1 + d_2 x + d_3 x^2$. There is a rule for how to remove superfluous terms, which combines easily with Rule I:

**Rule II.** Terms removed from Rule I appear in groups of related atoms

$$B(x), \quad xB(x), \quad \ldots, \quad x^m B(x),$$

where $B(x)$ is a base atom, that is, an atom not containing a power of $x$. Rule I removes the first $k$ of these atoms from the trial solution. Rule II removes the last $n - k$ of these atoms. The ones removed are called superfluous atoms.
An Illustration

Assume the differential equation has order $n = 2$ and the trial solution contains a sub-list of related atoms

$$e^{2x}, xe^{2x}, x^2e^{2x}, x^3e^x.$$  

Example 1

Assume $e^{2x}$ is not a solution of the homogeneous equation.

Then Rule I removes no atoms ($k = 0$) and Rule II removes the last 2 atoms ($n - k = 2 - 0 = 2$), resulting in the revised shorter atom sub-list

$$e^{2x}, xe^{2x}.$$  

Example 2

Assume $e^{2x}$ is a solution of the homogeneous equation.

Then Rule I removes atom $e^{2x}$ ($k = 1$) from the start of the list and Rule II removes $x^3e^{2x}$ from the end of list ($n - k = 2 - 1 = 1$), resulting in the revised sub-list

$$xe^{2x}, x^2e^{2x}.$$
Observations

• Rule I and Rule II together imply that exactly \( n \) atoms are removed from every complete sub-list of related atoms in the original trial solution.

• The \( n \) atoms are removed from the two ends, killing \( k \) from the beginning of the list and \( n - k \) from the end of the list.

• Substitution of the trial solution into the differential equation creates a the system of linear algebraic equations for the undetermined coefficients \( d_1, d_2, d_3, \ldots \), in which every symbol \( d_j \) appears! There are no free variables and the total number of atoms used in \( y \) cannot be reduced.

• The system of equations has the least possible dimension and a unique solution for the undetermined coefficients.
A Shortcut

Building the atom list from $g(x) = x^n f(x)$ requires subsequent removal of $n$ atoms from each sub-list of related atoms. Building a short atom list from $f(x)$ requires a subsequent append of atoms to each sub-list of related atoms. The second method, which requires less writing, is a shortcut recommended after learning the basic method of removing atoms.

The idea for appending the atoms is the realization that the factor $x^n$ used in $g(x) = x^n f(x)$ causes $n$ extra atoms to appear in a sub-list of related atoms. Here are the facts:

- If the first atom in the sublist, base atom $B$, is a solution of the homogeneous differential equation, then it is removed. This causes the first of the $n$ appended atoms to be kept.
- If the first two atoms $B, xB$ are solutions of the homogeneous differential equation, then both are removed. This causes the first two of the $n$ appended atoms to be kept.
- If the first three atoms $B, xB, x^2 B$ are solutions of the homogeneous differential equation, then all three are removed. This causes the first three of the $n$ appended atoms to be kept.
A Shortcut for Correction Rule II

Let a sub-list of related atoms be constructed from \( f(x) \) instead of \( g(x) = x^n f(x) \).
Each removal of an atom from the left causes an append of a related atom on the right.

**An Example for** \( f(x) = 11.578x^3e^x + 22.1 \cos 2x \)

Consider the sub-list constructed from atom \( x^3e^x \). The other atom \( \cos 2x \) is treated similarly. Assume \( n = 3 \) and \( e^x, xe^x \) are homogeneous DE solutions.

<table>
<thead>
<tr>
<th>Long sub-list from ( x^n f(x) )</th>
<th>( e^x )</th>
<th>( xe^x )</th>
<th>( x^2e^x )</th>
<th>( x^3e^x )</th>
<th>( x^4e^x )</th>
<th>( x^5e^x )</th>
<th>( x^6e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short sub-list from ( f(x) )</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
</tr>
<tr>
<td>Remove one on the left</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
<td>( x^4e^x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Append one on the right</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
</tr>
<tr>
<td>Remove one more from the left</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
<td>( x^4e^x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Append one more on the right</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
<td>( x^4e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
</tr>
<tr>
<td>Corrected list</td>
<td>( e^x )</td>
<td>( xe^x )</td>
<td>( x^2e^x )</td>
<td>( x^3e^x )</td>
<td>( x^4e^x )</td>
<td>( x^5e^x )</td>
<td>( x^6e^x )</td>
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