

**Example.** Three Possibilities with Symbol  $k$

Determine all values of the symbol  $k$  such that the system below has one of the *Three Possibilities*

- (1) No solution,
- (2) Infinitely many solutions,
- (3) A unique solution.

Display all solutions found.

$$\begin{aligned}x + ky &= 2, \\(2 - k)x + y &= 3.\end{aligned}$$

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The solution of this problem involves construction of three frame sequences, the last frame of each resulting in one classification among the Three Possibilities:

- (1) No solution, (2) Unique solution, (3) Infinitely many solutions.

The plan, for each of the three possibilities, is to obtain a triangular system by application of swap, multiply and combination rules. The three possibilities are detected by (1) A signal equation “ $0 = 1$ ,” (2) One or more free variables, (3) Zero free variables.

## Shared Frames

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The three expected frame sequences share these initial frames:

$$\begin{array}{rcl} x + & ky = & 2, \\ (2 - k)x + & y = & 3. \end{array}$$

Frame 1.

Original system.

$$\begin{array}{rcl} x + & ky = & 2, \\ 0 + [1 + k(k - 2)]y = & 2(k - 2) + & 3. \end{array}$$

Frame 2.

combo (1, 2, k-2)

$$\begin{array}{rcl} x + & ky = & 2, \\ 0 + & (k - 1)^2 y = & 2k - 1. \end{array}$$

Frame 3.

Simplify.

At this point, we identify the values of  $k$  that split off into the three possibilities.

## Analysis of the Frames

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$$\begin{array}{r} x + ky = 2, \\ 0 + (k - 1)^2 y = 2k - 1. \end{array}$$

Frame 3.

Simplify.

- There will be a signal equation if the second equation of Frame 3 has no variables, but the resulting equation is not “ $0 = 0$ .” This happens exactly for  $k = 1$ . The resulting signal equation is “ $0 = 1$ .” We conclude that one of the three frame sequences terminates with the *no solution case*. This frame sequence corresponds to  $k = 1$ .
- Otherwise,  $k \neq 1$ . For these values of  $k$ , there are zero free variables, which implies a unique solution.
- A by-product of the analysis is that the *infinitely many solutions* case never occurs!

## The Conclusion

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The three frame sequences reduce to two frame sequences. One sequence gives no solution and the other sequence gives a unique solution.

## The Three Answers

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- (1) There is no solution only for  $k = 1$ .
- (2) Infinitely many solutions never occur for any value of  $k$ .
- (3) For  $k \neq 1$ , there is a unique solution

$$\begin{aligned}x &= 2 - k(2k - 1)/(k - 1)^2, \\y &= (2k - 1)/(k - 1)^2.\end{aligned}$$

