Math 2250, Numerical Methods  
Maple Project Sample Solution  
Spring 2010

References: Code in maple appears in 2250mapleL4-sample-S2010.txt at URL http://www.math.utah.edu/~gustafso/. This document: 2250mapleL4-sample-S2010.pdf. Other related and required documents are available at the web site:

- Numerical Solution of First Order DE (typeset, 19 pages, 220k pdf). A resource similar to the textbook, with maple examples and deeper detail. It is for a second reading, in case Edwards-Penney left too many questions unanswered.
- Numerical DE coding hints, 2250numerical-hints.txt, TEXT Document (1 page, 2k). A modified portion of this document is appended here, for completeness.
- The web copy 2250mapleL4-sample-S2010.txt of the text in this document is suited for mouse copy and paste of maple code segments.

Problem ER-2. (E & P Exercise 2.6-36, Symbolic Solution)
The exact symbolic solution of the Logistic problem \( y' = 0.02225y - 0.0003y^2, \; y(0) = 25 \) is

\[
y(x) = \frac{2225}{30 + 59e^{-89x/4000}}
\]

Using textbook techniques, Chapter 2, derive the answer. Then check the answer in maple.

Solution.  
Derivation Details. The differential equation is a Verhulst-Logistic equation, studied in Section 2.1, appearing as equation (6):

\[
\frac{dy}{dx} = ky(M - y), \quad kM = 0.02225, \quad k = 0.0003.
\]

The unique solution \( y(x) \) with \( y(0) = y_0 \) is given by equation (7):

\[
y(x) = \frac{My_0}{y_0 + (M - y_0)e^{-kMx}}.
\]

The fraction will be multiplied top and bottom by the factor \( k/y_0 \), to obtain

\[
y(t) = \frac{k/y_0}{k/y_0} \cdot \frac{My_0}{y_0 + (M - y_0)e^{-kMx}}
\]

\[
= \frac{kM}{k + (kM/y_0 - k)e^{-kMx}}
\]

\[
= \frac{0.02225}{0.02225 + (0.02225/25 - 0.0003)e^{-0.02225x}}
\]

\[
= \frac{2225}{100000 + 0.0003 + (0.02225/25 - 0.0003)e^{-0.02225x}}
\]

\[
= \frac{2225}{30 + 59e^{-89x/4000}}.
\]

Answer Check in Maple.

```maple
# Check the exact symbolic solution
de:=diff(y(t),t)=0.02225*y(t) - 0.0003*y(t)^2;
ic:= y(0)=25;
dsolve({de,ic},y(t));
```

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Problem L4.1. (E & P Exercise 2.6-36)

Consider the initial value problem \( y' = 0.02225y - 0.0003y^2, \) \( y(0) = 25 \) with symbolic solution \( y(t) = \frac{2225}{30 + 59e^{-89t/4000}}. \)

Apply Euler’s method to find the numerical solution \( y(x) \) on \( x = 0 \) to \( x = 250. \) Write computer code to produce two dot tables. The first has \( n + 1 = 101 \) rows, \( h = 250/n = 2.5. \) The second has \( n + 1 = 201 \) rows, \( h = 250/n = 1.25. \) The computation should find the missing digits in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-approx, ( h = 2.5 )</td>
<td>25.00000000</td>
<td>45.0101????</td>
<td>61.2965????</td>
<td>69.4877????</td>
<td>72.6063????</td>
<td>73.6622????</td>
</tr>
<tr>
<td>y-approx, ( h = 1.25 )</td>
<td>25.00000000</td>
<td>45.0280????</td>
<td>61.2316????</td>
<td>69.4052????</td>
<td>72.5539????</td>
<td>73.6367????</td>
</tr>
<tr>
<td>actual ( y(x) )</td>
<td>25.00000000</td>
<td>45.04465339</td>
<td>61.16674082</td>
<td>69.32349992</td>
<td>72.50146404</td>
<td>73.61087799</td>
</tr>
<tr>
<td>Error(approx, actual)</td>
<td>0.0000%</td>
<td>0.03??%</td>
<td>0.10??%</td>
<td>0.11??%</td>
<td>0.07??%</td>
<td>0.03??%</td>
</tr>
</tbody>
</table>

Part I. Reproduce, by transcribing computer data, the table above, and fill in missing digits. For the percentage error with \( h = 250/200 = 1.25, \) use the equation

\[
\text{Error(approx, actual)} = 100 \times \frac{|\text{approx} - \text{actual}|}{|\text{actual}|}.
\]

Solution.

y-approx, \( h = 2.5, \) 25.00000000, 45.01012660, 61.29651142, 69.48777402, 72.60632272, 73.66229582.

y-approx, \( h = 1.25, \) 25.00000000, 45.02802159, 61.23165186, 69.40522495, 72.55394452, 73.63678526.

Symbolic \( y(x), \) 25.00000000, 45.04465339, 61.16674082, 69.32349992, 72.50146404, 73.61087799.

Error(approx, actual), \( h = 1.25, \) percentages 0.0, 0.03692291704, 0.1061214626, 0.1182504140, 0.07238540724, 0.03519489335.

Part II. Hand-check the first dot table for one step. The answer should be the same as line 2 of the first dot table (which has 101 lines). Assume the given symbolic solution is correct. Don’t repeat details already done in ER-2. Test the answers against the symbolic solution, as suggested in the table above.

Hand Check for Euler.

One step.
\( h = 2.5 \)
\( x_0 = 0 \)
\( y_0 = 25 \)
\( f(x,y) = 0.02225y - 0.0003y^2 \)
\( y_1 = y_0 + h f(x_0,y_0) \)
\( = 25 + 2.5 (0.02225 (25) - 0.0003 (25)^2) \)
\( = 25.921875 \)

Symbolic Solution Check.

The Euler answer and the symbolic answer agree to one digit.

Part III. Include an appendix of the computer code used.

```maple
# Now for the Euler code to make the dot table, error percentages and plot.
# Euler. Group 1, initialize.
f:=(x,y)->0.02225*y - 0.0003*y^2;
x0:=0:y0:=25:Dots:=[x0,y0]:n:=100:h:=250/n:
# Group 2, repeat n times. Euler's method
```
for i from 1 to n do
  Y:=y0+h*f(x0,y0);
  x0:=x0+h:y0:=Y:Dots:=Dots,[evalf(x0),evalf(y0)];
od:

# Group 3, display relevant dots and plot.
Exact:=x->2225/(30+59*exp(-89 *x/4000));
P:=unapply(evalf(100*abs(exact-approx)/abs(exact)),(exact,approx));
m:=n/5:X:=seq(1+m*j,j=0..n/m)]: # List of relevant indices
print("Dots"),seq(Dots[k],k=X);
print("Exact"),seq(Exact(Dots[k][1]),k=X);
print("Error"),seq(P(Exact(Dots[k][1]),Dots[k][2]),k=X);
#plot([Dots]);

### The output from this program:

"Dots"
[0, 25], [50., 45.01012660], [100., 61.29651142], [150., 69.48777402], [200., 72.60632272], [250., 73.66229582]

"Exact" 25, 45.04465339, 61.16674082, 69.32324992, 72.50146404, 73.61087799

"Error" 0., 0.07665014025, 0.2121587619, 0.2373288907, 0.1446297415, 0.06985085819

Problem L4.2. (E & P Exercise 2.6-36)

Consider the initial value problem \( y' = 0.02225 y - 0.0003 y^2 \), \( y(0) = 25 \) with symbolic solution \( y(t) = \frac{2225}{30 + 59e^{-89t/4000}} \).

Apply Heun's method to finds the numerical solution \( y(x) \) on \( x = 0 \) to \( x = 250 \). Write computer code to produce two dot tables. The first has \( n + 1 = 101 \) rows, \( h = 250/200 = 1.25 \). The second has \( n + 1 = 201 \) rows, \( h = 250/200 = 1.25 \). The computation should find the missing digits in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-approx, ( h = 2.5 )</td>
<td>25.000000000</td>
<td>45.0419????</td>
<td>61.1624????</td>
<td>69.3195????</td>
<td>72.4992????</td>
<td>73.6098????</td>
</tr>
<tr>
<td>y-approx, ( h = 1.25 )</td>
<td>25.000000000</td>
<td>45.0439????</td>
<td>61.1656????</td>
<td>69.3223????</td>
<td>72.5009????</td>
<td>73.6106????</td>
</tr>
<tr>
<td>actual ( y(x) )</td>
<td>25.000000000</td>
<td>45.04465339</td>
<td>61.16674082</td>
<td>69.32324992</td>
<td>72.50146404</td>
<td>73.61087799</td>
</tr>
<tr>
<td>Error(approx,actual)</td>
<td>0.0000%</td>
<td>0.0017??%</td>
<td>0.0017??%</td>
<td>0.0017??%</td>
<td>0.0007??%</td>
<td>0.0007??%</td>
</tr>
</tbody>
</table>

Part I. Reproduce, by transcribing computer data, the table above, and fill in missing digits. For the percentage error with \( h = 250/200 = 1.25 \), use the equation

\[
Error(approx, actual) = 100 \frac{|approx - actual|}{|actual|}.
\]

Solution.

y-approx, \( h = 2.5 \), 25, 45.04191584, 61.16246299, 69.31954666, 72.49927181, 73.60981811.
y-approx, \( h = 1.25 \), 25, 45.04396719, 61.16567946, 69.32233642, 72.50092484, 73.61061773.
Symbolic \( y(x) \), 25, 45.04465339, 61.16674082, 69.32324992, 72.50146404, 73.61087799.
Error(approx,actual), \( h = 1.25 \), percentages 0.0, 0.001523377245, 0.001735191357, 0.001317739721, 0.0007437091197, 0.0003535618744.

Part II. Hand-check the first dot table for one step. The answer should be the same as line 2 of the first dot table (which has 101 lines). Assume the given symbolic solution is correct. Don’t repeat details already done in ER-2. Test the answers against the symbolic solution, as suggested in the table above.

Hand Check for Heun.

One step.
\( h=2.5 \)
x0 = 0
y0 = 25
\( f(x,y) = 0.02225 y - 0.0003 y^2 \)
y1 = y0 + h f(x0,y0) = 25 + 2.5 (0.02225 (25) - 0.0003 (25)^2) = 25.922875
\[ y_2 = y_0 + h(f(x_0, y_0) + f(x_0+h, y_1))/2 = 25 + 2.5 \left(\frac{0.02225(25) - 0.0003(25)^2}{2}\right) + 2.5 \left(\frac{0.02225(25.921875) - 0.0003(25.921875)^2}{2}\right) = 25.92991080 \]

Dots[1] = [0, 25], Dots[2] = [2.500000000, 25.92991080]. Answer checks.

Symbolic Solution Check.
The Heun answer and the symbolic answer agree to two digits.

Part III. Include an appendix of the computer code used.

# Now for the Heun code to make the dot table, error percentages and plot.
# Heun. Group 1, initialize.
\[ f := (x, y) \rightarrow 0.02225\, y - 0.0003\, y^2; \]
\[ x_0 := 0; y_0 := 25; \text{Dots} := [x_0, y_0]; n := 100; h := 250/n; \]
# Group 2, repeat n times. Heun’s method
for i from 1 to n do
Y1 := y0 + h * f(x0, y0);
Y := y0 + h * (f(x0, y0) + f(x0 + h, Y1))/2;
x0 := x0 + h; y0 := Y;
Dots := Dots, [evalf(x0), evalf(y0)];
end do:
# Group 3, display relevant dots and plot.
\[ \text{Exact} := x \rightarrow \frac{2225}{30 + 59 \, e^{-89\, x/4000}}; \]
\[ P := \text{unapply}(\text{evalf}(100*\text{abs}(\text{exact}-\text{approx})/\text{abs}(\text{exact})), (\text{exact}, \text{approx})); \]
m := n/5:
x := \{seq(1 + m*j, j = 0 .. n/m)\}; # List of relevant indices
print("Dots"), seq(Dots[k][1], k = X);
print("Exact"), seq(Exact(Dots[k][1]), k = X);
print("Error"), seq(P(Exact(Dots[k][1]), Dots[k][2]), k = X);
#plot([Dots]);

### The output from this program:

<table>
<thead>
<tr>
<th>x</th>
<th>0.0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-approx, h = 2.5</td>
<td>25.000000000</td>
<td>45.04465339</td>
<td>61.16674082</td>
<td>69.32324992</td>
<td>72.50146404</td>
<td>73.61087799</td>
</tr>
<tr>
<td>y-approx, h = 1.25</td>
<td>25.000000000</td>
<td>45.04465339</td>
<td>61.16674082</td>
<td>69.32324992</td>
<td>72.50146404</td>
<td>73.61087799</td>
</tr>
<tr>
<td>actual y(x)</td>
<td>25.000000000</td>
<td>45.04465339</td>
<td>61.16674082</td>
<td>69.32324992</td>
<td>72.50146404</td>
<td>73.61087799</td>
</tr>
<tr>
<td>Error(approx, actual)</td>
<td>0.000000%</td>
<td>0.000000%</td>
<td>0.000000%</td>
<td>0.000000%</td>
<td>0.000000%</td>
<td>0.000000%</td>
</tr>
</tbody>
</table>

Problem L4.3. (E & P Exercise 2.6-36)
Consider the initial value problem \( y' = 0.02225y - 0.0003y^2 \), \( y(0) = 25 \) with symbolic solution \( y(t) = \frac{2225}{30 + 59e^{-89\, t/4000}} \).
Apply the RK4 method to finds the numerical solution \( y(x) \) on \( x = 0 \) to \( x = 250 \). Write computer code to produce two dot tables. The first has \( n+1 = 101 \) rows, \( h = 250/n = 2.5 \). The second has \( n+1 = 201 \) rows, \( h = 250/n = 1.25 \). The computation should find the missing digits in the table below.

<table>
<thead>
<tr>
<th>x</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>y-approx, h = 2.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y-approx, h = 1.25</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>actual y(x)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error(approx, actual)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Part I. Reproduce, by transcribing computer data, the table above, and fill in missing digits. For the percentage error with \( h = 250/200 = 1.25 \), use the equation
\[
\text{Error(approx, actual)} = 100 \frac{|\text{approx} - \text{actual}|}{|\text{actual}|}.
\]

Solution.
y-approx, h = 2.5, 25.0, 45.04465322, 61.16674048, 69.32324952, 72.50146380, 73.61087789.
y-approx, h = 1.25, 25.0, 45.04465348, 61.16674086, 69.32324992, 72.50146405, 73.61087799.
Symbolic $y(x)$, 25, 45.04465339, 61.16674082, 69.32324992, 72.50146404, 73.61087799.
Error(approx,actual), $h = 1.25$, percentages 0.0, 0.3774032814e-6, 0.5558576368e-6, 0.5770069933e-6, 0.3310277981e-6, 0.1358494868e-6.

Part II. Assume the given symbolic solution is correct. Don’t repeat details already done in ER-2. Test the answers against the symbolic solution, as suggested in the table above.

Symbolic Solution Check.
The RK4 answer and the symbolic answer **agree to six digits**.

Part III. Include an appendix of the computer code used.

```plaintext
# Now for the RK4 code to make the dot table, error percentages and plot.
# RK4. Group 1, initialize.
f:=(x,y)->0.02225 *y - 0.0003*y^2;
x0:=0:y0:=25:Dots:=[x0,y0]:n:=100:h:=250/n:
# Group 2, repeat n times. RK4 method.
for i from 1 to n do
  k1:=h*f(x0,y0):
k2:=h*f(x0+h/2,y0+k1/2):
k3:=h*f(x0+h/2,y0+k2/2):
k4:=h*f(x0+h,y0+k3):
  Y:=y0+(k1+2*k2+2*k3+k4)/6:
x0:=x0+h:y0:=Y:Dots:=Dots,[evalf(x0),evalf(y0)];
od:
# Group 3, display relevant dots and plot.
Exact:=x->2225/(30+59*exp(-89 *x/4000));
P:=unapply(evalf(100*abs(exact-approx)/abs(exact)),(exact,approx)):
m:=n/5:X:= [seq(1+m*j,j=0..n/m)]: # List of relevant indices
print("Dots"),seq(Dots[k][1],k=X);
print("Exact"),seq(Exact(Dots[k][1]),k=X);
print("Error"),seq(P(Exact(Dots[k][1]),Dots[k][2]),k=X);
plot(Dots);
```

### The output from this program:

```
"Dots"
[0, 25], [50., 45.04465322], [100., 61.16674048],
[150., 69.32324952], [200., 72.50146380], [250., 73.61087799]
"Exact"
25, 45.04465339, 61.16674082, 69.32324992, 72.50146404, 73.61087799
"Error"
0.0, .3774032814e-6, .5558576368e-6, .5770069933e-6, .3310277981e-6, .1358494868e-6
```

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# Warning: These snips of code made for \( y' = 1 - x - y \), \( y(0) = 3 \).

# Code computes approx values for \( y(0.1) \) to \( y(1) \).
# 'Dots' is the list of dots for connect-the-dots graphics.

# ========================================
# Euler. Group 1, initialize.

\( f := \langle x, y \rangle \rightarrow 1 - x - y; \)
\( x_0 := 0; y_0 := 3; h := 0.1; \)
\( \text{Dots} := [x_0, y_0]; n := 200; \)

# Group 2, repeat n times. Euler’s method

for \( i \) from 1 to \( n \) do
\( Y := y_0 + h \cdot f(x_0, y_0); \)
\( x_0 := x_0 + h; y_0 := Y; \)
\( \text{Dots} := \text{Dots}, [x_0, y_0]; \)
od:

# Group 3, display relevant dots and plot.

\( \text{Exact} := x \rightarrow 2 - x + \exp(-x); \)
\( P := \text{unapply}(\text{evalf}(100 \cdot \text{abs}(\text{exact} - \text{approx})/\text{abs}(\text{exact})), (\text{exact}, \text{approx})); \)
\( m := 40; X := [\text{seq}(i + m \cdot j, j = 0 .. \lfloor n / m \rfloor)]; \# \text{List of relevant indices} \)
print("Dots"), \text{seq}([X[k], k = X];
print("Exact"), \text{seq}(\text{Exact}(X[k])[1], k = X);
print("Error"), \text{seq}(P(\text{Exact}(X[k])[1]), \text{Dots}[k][2], k = X);
plot([Dots]);

# ========================================
# Heun. Group 1, initialize.

\( f := \langle x, y \rangle \rightarrow 1 - x - y; \)
\( x_0 := 0; y_0 := 3; h := 0.1; \)
\( \text{Dots} := [x_0, y_0]; n := 200; \)

# Group 2, repeat n times. Heun method

for \( i \) from 1 to \( n \) do
\( Y_1 := y_0 + h \cdot f(x_0, y_0); \)
\( Y := y_0 + h \cdot (f(x_0, y_0) + f(x_0 + h, Y_1))/2; \)
\( x_0 := x_0 + h; y_0 := Y; \)
\( \text{Dots} := \text{Dots}, [x_0, y_0]; \)
od:

# Group 3, display relevant dots and plot.

\( \text{Dots}[1], \text{Dots}[2], \text{seq}(\text{Dots}[1 + 40 \cdot j], j = 1 .. \lfloor n/40 \rfloor); \)
plot([Dots]);

# ========================================
# RK4. Group 1, initialize.

\( f := \langle x, y \rangle \rightarrow 1 - x - y; \)
\( x_0 := 0; y_0 := 3; h := 0.1; \)
\( \text{Dots} := [x_0, y_0]; n := 100; \)

# Group 2, repeat n times. RK4 method

for \( i \) from 1 to \( n \) do
\( k_1 := h \cdot f(x_0, y_0); \)
\( k_2 := h \cdot f(x_0 + h/2, y_0 + k_1/2); \)
\( k_3 := h \cdot f(x_0 + h/2, y_0 + k_2/2); \)
\( k_4 := h \cdot f(x_0 + h, y_0 + k_3); \)
\( Y := y_0 + (k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4)/6; \)
\( x_0 := x_0 + h; y_0 := Y; \)
\( \text{Dots} := \text{Dots}, [x_0, y_0]; \)
od:

# Group 3, display some dots and plot.

\( \text{Dots}[1], \text{Dots}[2], \text{Dots}[101]; \)
plot([Dots]);