Differential Equations and Linear Algebra

2250-4 at 10:15am on 17 Dec 2010

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on **WEBCT**. Recording errors are reported by email.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

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Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

Ch3. (Linear Systems and Matrices) Complete all problems.

[10%] **Ch3(a)**: Check the correct box. Incorrect answers lose all credit.
Part 1. [5%]: ☐ True or ☐ False:
If the 3×3 matrices A and B are upper triangular, then the product AB is triangular.
Part 2. [5%]: ☐ True or ☐ False:
If a 3×3 matrix A has determinant zero, then for all vectors $\mathbf{b}$,
the equation $A\mathbf{x} = \mathbf{b}$ has infinitely solutions $\mathbf{x}$.

Answer: True. False.

[40%] **Ch3(b)**: Determine which values of k correspond to infinitely many solutions for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer: There is a unique solution for $\det(A) \neq 0$, which is equivalent to $k \neq 2$ and $k \neq 3$. Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & 3-k & k-1 \end{pmatrix}$. Then (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 2$.

[30%] **Ch3(c)**: Define matrix A and vector $\mathbf{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $A\mathbf{x} = \mathbf{b}$.

Answer: $x_2 = \Delta_2/\Delta$, $\Delta_2 = \det \begin{pmatrix} -2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} = 20$, $\Delta = \det(A) = 4$, $x_3 = 5$.

[20%] **Ch3(d)**: Assume the transpose of A is $A^T = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the transpose of the inverse of A.

Answer: Then $(A^{-1})^T = (A^T)^{-1} = \left(\begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}\right)^{-1} = \frac{1}{8}\begin{pmatrix} 4 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{4} \end{pmatrix}$.

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Check the independence tests which apply to prove that 1, x, $x\sin(x)$ are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Rank test**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Atom test**	Any finite set of distinct atoms is independent.
☐	**Wronskian test**	The Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Sample test**	Define $\mathbf{v}(x) = \begin{pmatrix} f_1(x), f_2(x), f_3(x) \end{pmatrix}$. If A has rows $\mathbf{v}(x_1)$, $\mathbf{v}(x_2)$, $\mathbf{v}(x_3)$ and $\det(A) \neq 0$, then the functions are independent.

Answer: The last three apply to the given functions, while the others apply only to fixed vectors.

[20%] **Ch4(b)**: Give an example of a 3×4 matrix A of rank 2 with exactly one zero entry. Include an explanation of why your example has rank 2.

Answer: Let A initially be the matrix of all ones, then set $a_{11} = 0$.

[30%] **Ch4(c)**: Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\mathbf{x} = \mathbf{0}$. Apply the kernel theorem. This is theorem 2 in section 4.2 of Edwards-Penney. Then S is a subspace of $\mathcal{R}^3$.

[30%] **Ch4(d)**: Apply an independence test to the vectors below. Report **independent** or **dependent**. Details count.

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

Answer: Independent. The rank of the augmented matrix of the three vectors is 3. Details should compute the rank or the number of pivot columns. The book has a determinant shortcut, which is OK, but it should be explained. There is no such thing as the determinant of a 4×3 matrix.

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = 15x^2 + 17e^{-x} + 100x\sin 2x$ is a solution.

Answer: The atoms x^2, e^{-x}, $x\sin 2x$ correspond to roots $0, 0, 0, -1, 2i, -2i, 2i, -2i$. The factor theorem implies the characteristic polynomial should be $r^3(r+1)(r^2+4)^2$.

[20%] **Ch5(b)**: Determine a basis of solutions of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$r(r^2 - r)^2((r+2)^2 + 81)^2 = 0.$$

Answer: The roots are $0, 0, 0, 1, 1, -2 \pm 9i, -2 \pm 9i$. By Euler's theorem, a basis is the set of 9 atoms for these roots: $1, x, x^2, e^x, xe^x, e^{2x}\cos(9x), e^{2x}\sin(9x), xe^{2x}\cos(9x), xe^{2x}\sin(9x)$.

[30%] **Ch5(c)**: Find the steady-state periodic solution for the equation

$$x'' + 2x' + 9x = 34\cos(t).$$

It is known that this solution equals the undetermined coefficients solution for a particular solution $x_p(t)$.

Answer: Use undetermined coefficients trial solution $x = d_1\cos t + d_2\sin t$. Substitute and solve for $d_1 = 4$, $d_2 = 1$.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 3x^2 + 4\cos 2x + 5xe^x$$

Answer: Let $f(x) = 3x^2 + 4\cos 2x + 5xe^x$. The atoms in f are x^2, $\cos 2x$, xe^x. The complete set of distinct atoms appearing in the derivatives $f, f', f'', \ldots$ is $1, x, x^2, \cos 2x, \sin 2x, e^x, xe^x$. There are 7 atoms in this list. A theorem says that the shortest trial solution contains 7 atoms. Break the 7 atoms into four groups, each with the same base atom: group 1 $== 1, x, x^2$; group 2 $== \cos 2x$; group 3 $== \sin 2x$; group 4 $== e^x$, xe^x. Modify each group, by multiplication by x until the group contains no solution of the homogeneous equation $y^{(3)} - y^{(2)} = 0$. Then the four groups are replaced by group $1^* == x^2, x^3, x^4$; group $2^* == \cos 2x$; group $3^* == \sin 2x$; group $4^* == xe^x$, x^2e^x. The shortest trial solution is a linear combination of these last seven atoms.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[20%] **Ch6(a)**: Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer: The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues, taken in the same order.

[40%] **Ch6(b)**: Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 4 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer: The characteristic polynomial is $\det(A - rI) = (-r)(4-r)(r-2)^2$. The eigenvalues are $0, 2, 2, 4$. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a 3×3 determinant, which can be expanded by the cofactor method along column 3.

[40%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$.
To save time, **don't find the eigenvector for** $\lambda = 0$.

Answer: Because $A - 2I = \begin{pmatrix} -2 & -2 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ has last frame $B = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then there is only one eigenpair for $\lambda = 2$, with eigenvector $\mathbf{v} = \begin{pmatrix} 1/2 \\ -1 \\ 1 \end{pmatrix}$. The usual eigenpair is $\left(2, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right)$.

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[50%] **Ch7(a)**: Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned} \frac{dx}{dt} &= 4x + 18y, \\ \frac{dy}{dt} &= 3x + y. \end{aligned}$$

Answer: Define $A = \begin{pmatrix} 4 & 18 \\ 3 & 1 \end{pmatrix}$. The eigenvalues $-5, 10$ are roots of the characteristic equation $\det(A - rI) = (r+5)(r-10) = 0$. By Cayley-Hamilton-Zeibur, $x(t) = c_1 e^{-5t} + c_2 e^{10t}$. Using the first differential equation $x' = 4x + 18y$ implies $18y(t) = x' - 4x = -9c_1 e^{-5t} + 6c_2 e^{10t}$. Then $y(t) = -\frac{1}{2}c_1 e^{-5t} + \frac{1}{3}c_2 e^{10t}$. Eigenanalysis would use the eigenpairs $\left(-5, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right), \left(10, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$ to report solution $\mathbf{u}(t) = c_1 \mathbf{v}_1 e^{-5t} + c_2 \mathbf{v}_2 e^{10t}$.

[50%] **Ch7(b)**: Define

$$A = \begin{pmatrix} 5 & -1 & -1 \\ 0 & 3 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

The eigenvalues of A are 3, 3, 6. Apply the eigenanalysis method, which requires eigenvalues and eigenvectors, to solve the differential system $\mathbf{u}' = A\mathbf{u}$. Show all eigenanalysis steps and display the differential equation answer $\mathbf{u}(t)$ in vector form.

Answer: Form $A_1 = A - (3)I = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$. Then $\mathbf{rref}(A_1) = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The last frame algorithm implies the general solution of $A_1\mathbf{x} = \mathbf{0}$ is $x_1 = t_1/2 + t_2/2$, $x_2 = t_1$, $x_3 = t_2$. Take the partials on symbols t_1, t_2 to obtain the eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$. Repeat with $A_2 = A - (6)I = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -3 & 0 \\ -2 & 1 & -2 \end{pmatrix}$. Then $\mathbf{rref}(A_2) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The last frame algorithm implies the general solution of $A_2\mathbf{x} = \mathbf{0}$ is $x_1 = -t_1$, $x_2 = 0$, $x_3 = t_1$. Take the partial on symbol t_1 to obtain the eigenvector $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. The general solution of $\mathbf{u}' = A\mathbf{u}$ is $\mathbf{u}(t) = c_1 e^{3t}\mathbf{v}_1 + c_2 e^{3t}\mathbf{v}_2 + e^{6t}\mathbf{v}_3$.

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Ch8. (Matrix Exponential) Complete all problems.

[40%] **Ch8(a)**: Using any method in the lectures or the textbook, display the matrix exponential e^{Bt}, for

$$B = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}.$$

Answer: We find a fundamental matrix $Z(t)$. It is done by solving the matrix system $\mathbf{u}' = B\mathbf{u}$ with Ziebur's shortcut. Then $x = c_1 \cos 3t + c_2 \sin 3t$, $3y = x' = -3c_1 \sin 3t + 3c_2 \cos 3t$, and finally $y = -c_1 \sin 3t + c_2 \cos 3t$. Write $\mathbf{u} = Z(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $Z(t) = \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}$. Because $Z(0)$ is the identity, then

$$e^{Bt} = Z(t) = \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}.$$

Putzer's formula can be used, $e^{Bt} = \text{Real part}\left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(B - \lambda_1 I)\right)$. The eigenvalues $\lambda_1 = 3i$, $\lambda_2 = -3i$ are found from $\det(B - \lambda I) = \lambda^2 + 9 = 0$. A third method is to use the formula $e^{Bt} = \mathcal{L}^{-1}((sI - B)^{-1})$, to find the answer in a series of Laplace steps.

[30%] **Ch8(b)**: Consider the 2×2 system

$$\begin{aligned} x' &= 3x, \\ y' &= -y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Solve the system as a matrix problem $\mathbf{u}' = A\mathbf{u}$ for $\mathbf{u}$, using the matrix exponential e^{At}.

Answer: Let $A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$. The answer for e^{At} can be obtained quickly from the theorem $e^{\mathbf{diag}(a,b)t} = \mathbf{diag}(e^{at}, e^{bt})$, giving the answer $e^{At} = \mathbf{diag}(e^{3t}, e^{-t})$. Then use $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$, which implies $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{-t} \end{pmatrix}$.

[30%] **Ch8(c)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$\begin{aligned} x' &= 3x + 3, \\ y' &= -y + 1. \end{aligned}$$

Answer: Variation of parameters is $\mathbf{u}_p(t) = e^{At} \int_0^t e^{-As} \begin{pmatrix} 3 \\ 1 \end{pmatrix} ds$. Then $e^{At} = \mathbf{diag}(e^{3t}, e^{-t})$ from the previous problem. Substitute $t \to -s$ to obtain $e^{-As} = \mathbf{diag}(e^{-3s}, e^{s})$. The integration step is

$$\int_0^t \mathbf{diag}(e^{-3s}, e^{s}) \begin{pmatrix} 3 \\ 1 \end{pmatrix} ds = \int_0^t \begin{pmatrix} 3e^{-3s} \\ e^{s} \end{pmatrix} ds = \begin{pmatrix} 1 - e^{-3t} \\ e^{t} - 1 \end{pmatrix}.$$

Finally, $\mathbf{u}_p(t) = \mathbf{diag}(e^{3t}, e^{-t}) \begin{pmatrix} 1 - e^{-3t} \\ e^{t} - 1 \end{pmatrix} = \begin{pmatrix} e^{3t} - 1 \\ 1 - e^{-t} \end{pmatrix}$.

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the equilibrium $\mathbf{u} = \mathbf{0}$ is stable or unstable. Then classify the equilibrium point $\mathbf{u} = \mathbf{0}$ as a saddle, center, spiral or node.

$$\mathbf{u}' = \begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix} \mathbf{u}$$

Answer: The eigenvalues of A are roots of $r^2 + 2r + 5 = (r+1)^2 + 4 = 0$, which are complex conjugate roots $-1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^{-t}\cos 2t$, $e^{-t}\sin 2t$ have limit zero at $t = \infty$, therefore the system is stable at $t = \infty$. So it must be a spiral [centers have no exponentials]. Report: **stable spiral**.

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y^2 - y + 9, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 3$, $y = 3$. Compute the Jacobian matrix $A = J(3,3)$ of the linearized system at this equilibrium point.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} 1 & -2y-1 \\ 4x-2y & -2x \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} 1 & -7 \\ 6 & -6 \end{pmatrix}$.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= 4x + 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{aligned}$$

At equilibrium point $x = 3$, $y = -3$, the Jacobian matrix is $A = J(3,-3) = \begin{pmatrix} -2 & 4 \\ 3 & 3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\mathbf{u} = \mathbf{0}$ for the **linear** dynamical system $\mathbf{u}' = A\mathbf{u}$.

(2) Apply a theorem to classify $x = 3$, $y = -3$ as a saddle, center, spiral or node for the **nonlinear** dynamical system. Discuss all details of the application of the theorem.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} 4-2x & 4 \\ 3 & 3 \end{pmatrix}$. Then $A = J(3,-3) = \begin{pmatrix} -2 & 4 \\ 3 & 3 \end{pmatrix}$. The eigenvalues of A are found from $r^2 - r - 18 = 0$, giving two different real roots a, b, one positive and one negative. The atoms are e^{at}, e^{bt}. Because the roots are real, rotation does not happen, and the classification must be a saddle or a node. One atom has limit zero at $t = \infty$, and the other limit zero at $t = -\infty$. Therefore it cannot be a node. The classification is a saddle, and we report **unstable saddle** for the linear problem $\mathbf{u}' = A\mathbf{u}$ at equilibrium $\mathbf{u} = \mathbf{0}$. Theorem 2 in 9.2 applies to say that the same is true for the nonlinear system: **unstable saddle** at $x = 3$, $y = -3$.

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Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\dfrac{1}{s^3}$	$\dfrac{1}{s-4}$	$\dfrac{s}{s^2+4}$	$\dfrac{2}{s^2+1}$	$\dfrac{e^{-s}}{s+1}$

$f(t)$	t	te^t	$t\cos 2t$	$t(1+e^t)$	$e^t \sin t$
$\mathcal{L}(f(t))$					

Answer: First table left to right: $\dfrac{t^2}{2}$, e^{4t}, $\cos 2t$, $2\sin t$, $e^{1-t}\,\mathbf{step}(t-1)$. Function $\mathbf{step}(t)$ is the unit step $u(t)$ of the textbook, $\mathbf{step}(t) = 1$ for $t \geq 0$, zero elsewhere. Second table left to right: $\dfrac{1}{s^2}$, $\dfrac{1}{(s-1)^2}$, $\dfrac{-d}{ds}\dfrac{s}{s^2+4} = \dfrac{s^2-4}{(s^2+4)^2}$, $\dfrac{1}{s^2} + \dfrac{1}{(s-1)^2}$, $\dfrac{1}{(s-1)^2+1}$.

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for the pulse $f(t) = t$ on $2 \leq t < 3$, $f(t) = 0$ otherwise.

Answer: Define $\mathbf{step}(t)$ to be the unit step. Use $f(t) = t\,\mathbf{step}(t-2) - t\,\mathbf{step}(t-3)$, and the second shifting theorem. Then $\mathcal{L}(t\,\mathbf{step}(t-2)) = e^{-2s}\mathcal{L}(t|_{t\to t+2}) = e^{-2s}\mathcal{L}(t+2) = e^{-2s}(\frac{1}{s^2}+\frac{2}{s})$. Similarly, $\mathcal{L}(t\,\mathbf{step}(t-3)) = e^{-3s}\mathcal{L}(t|_{t\to t+3}) = e^{-3s}\mathcal{L}(t+3) = e^{-3s}(\frac{1}{s^2}+\frac{3}{s})$. The answer: $\mathcal{L}(f(t)) = e^{-2s}(\frac{1}{s^2}+\frac{2}{s}) - e^{-3s}(\frac{1}{s^2}+\frac{3}{s})$.

[30%] **Ch10(c)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) - 3x'(t) = 9e^{3t}, \quad x(0) = x'(0) = 0.$$

Answer: $x(t) = 1 - e^{3t} + 3te^{3t}$. The Laplace steps are: (1) $\mathcal{L}(x(t)) = \dfrac{9}{s(s-3)^2}$; (2) $\mathcal{L}(x(t)) = \dfrac{9}{s(s-3)^2} = \dfrac{A}{s} + \dfrac{B}{s-3} + \dfrac{C}{(s-3)^2} = \mathcal{L}(A + Be^{3t} + Cte^{3t})$ where by partial fractions the answers are $A = 1$, $B = -1$, $C = 3$. Lerch's theorem implies $x(t) = 1 - e^{3t} + 3te^{3t}$.

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