Name _____ **Lecture Meets at** _____

Differential Equations and Linear Algebra 2250

Midterm Exam 3 [12:25 lecture]
Version 24.11.2010

Scores
1.
2.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$\begin{aligned} x' &= 4x + y, \\ y' &= 3y, \\ x(0) &= 1, \quad y(0) = -1. \end{aligned}$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s-4)\mathcal{L}(x) + (-1)\mathcal{L}(y) = 1, \quad (0)\mathcal{L}(x) + (s-3)\mathcal{L}(y) = -1.$$

Eimination or Cramer's rule applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{1}{s-3}$. Then the backward table implies $x(t) = e^{3t}$.

(1b) [30%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{7s^2 - 6s + 3}{s^2(s-1)^2}.$$

Answer:

$\mathcal{L}(f(t)) = \frac{3}{s^2} + \frac{4}{(s-1)^2} = \mathcal{L}(3t + 4te^t)$ implies $f(t) = 3t + 4te^t$.

(1c) [30%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(f(t)) = \frac{d}{ds}\mathcal{L}(tf(t)) + \frac{24}{s^4}.$$

Answer:

Use the s-differentiation theorem and the backward Laplace table to get $\mathcal{L}(((-t)^2)f(t) - \mathcal{L}((-t)(t)f(t)) = 24\mathcal{L}(t^3/6)$. Lerch's theorem implies $f(t) = 2t$.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Chapter 10)** Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^{3t}\cos(5t)$	
$t^2e^{-2t/3}$	
$t\sin(2t)$	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{2}{s^2+4}$	$\sin 2t$
$\dfrac{s-4}{s^2-8s+20}$	
$\dfrac{3}{(3s+2)^2}$	
$\dfrac{2s}{s^2-6s+13}$	

Answer:

Forward: $\dfrac{s-3}{(s-3)^2+25}$, $\dfrac{2}{(s+2/3)^3}$, $-\dfrac{d}{ds}\dfrac{2}{s^2+4} = \dfrac{4s}{(s^2+4)^2}$.

Backward: $e^{4t}\cos(2t)$, $\dfrac{1}{3}te^{-2t/3}$, $2e^{3t}\cos(2t) + 3e^{3t}\sin(2t)$.

(2b) [40%] Find $\mathcal{L}(x(t))$, given $x(t) = (t-1)u(t-2) + u(t-1)$, where u is the unit step function, $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$.

Answer:

Use the second shifting theorem

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}\mathcal{L}(f(t)).$$

Write $x(t) = (t-2)u(t-2) + u(t-2) + u(t-1)$. Then $\mathcal{L}(x(t)) = \mathcal{L}((t-2)u(t-2)) + \mathcal{L}(u(t-2)) + \mathcal{L}(u(t-1)) = e^{-2s}\mathcal{L}(t) + e^{-2s}\mathcal{L}(1) + e^{-s}\mathcal{L}(1) = e^{-2s}\left(\dfrac{1}{s^2} + \dfrac{1}{s}\right) + e^{-s}\dfrac{1}{s}$.

Use this page to start your solution. Attach extra pages as needed.