Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Chapter 10) Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace’s method to solve the system for $x(t)$. Don’t waste time solving for $y(t)$!

\[
\begin{align*}
x' &= 4x + y, \\
y' &= 3y, \\
x(0) &= 1, \quad y(0) = -1.
\end{align*}
\]

Answer:

The Laplace resolvent equation $(sI - A)L(u) = u(0)$ can be written out to find a $2 \times 2$ linear system for unknowns $L(x(t))$, $L(y(t))$:

\[
(s - 4)L(x) + (-1)L(y) = 1, \quad (0)L(x) + (s - 3)L(y) = -1.
\]

Elimination or Cramer’s rule applies to this system to solve for $L(x(t)) = \frac{1}{s - 3}$. Then the backward table implies $x(t) = e^{3t}$.

(1b) [30%] Find $f(t)$ by partial fraction methods, given

\[
L(f(t)) = \frac{7s^2 - 6s + 3}{s^2(s - 1)^2}.
\]

Answer:

$L(f(t)) = \frac{3}{s} + \frac{4}{(s-1)^2} = L(3t + 4te^t)$ implies $f(t) = 3t + 4te^t$.

(1c) [30%] Solve for $f(t)$, given

\[
\frac{d^2}{ds^2}L(f(t)) = \frac{d}{ds}L(tf(t)) + \frac{24}{s^4}.
\]

Answer:

Use the $s$-differentiation theorem and the backward Laplace table to get $L((-t)^2)f(t) - L((-t)(t)f(t)) = 24L(t^3/6)$. Lerch’s theorem implies $f(t) = 2t$.
2. (Chapter 10) Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

<table>
<thead>
<tr>
<th>Forward Table</th>
<th>Backward Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( \mathcal{L}(f(t)) )</td>
</tr>
<tr>
<td>( t^3 )</td>
<td>( \frac{6}{s^4} )</td>
</tr>
<tr>
<td>( e^{3t} \cos(5t) )</td>
<td></td>
</tr>
<tr>
<td>( t^2 e^{-2t/3} )</td>
<td></td>
</tr>
<tr>
<td>( t \sin(2t) )</td>
<td></td>
</tr>
</tbody>
</table>

Answer:
Forward: \( \frac{s - 3}{(s - 3)^2 + 25}, \frac{2}{(s + 2/3)^3} - \frac{d}{ds} \frac{2}{s^2 + 4} = \frac{4s}{(s^2 + 4)^2} \).
Backward: \( e^{4t} \cos(2t), \frac{1}{3} te^{-2t/3}, 2e^{3t} \cos(2t) + 3e^{3t} \sin(2t) \).

(2b) [40%] Find \( \mathcal{L}(x(t)) \), given \( x(t) = (t - 1)u(t - 2) + u(t - 1) \), where \( u \) is the unit step function, \( u(t) = 1 \) for \( t \geq 0 \), \( u(t) = 0 \) for \( t < 0 \).

Answer:
Use the second shifting theorem
\[
\mathcal{L}(f(t - a)u(t - a)) = e^{-as} \mathcal{L}(f(t)).
\]

Write \( x(t) = (t - 2)u(t - 2) + u(t - 2) + u(t - 1) \). Then \( \mathcal{L}(x(t)) = \mathcal{L}((t - 2)u(t - 2)) + \mathcal{L}(u(t - 2)) + \mathcal{L}(u(t - 1)) = e^{-2s} \mathcal{L}(t) + e^{-2s} \mathcal{L}(1) + e^{-s} \mathcal{L}(1) = e^{-2s} \left( \frac{1}{s^2} + \frac{1}{s} \right) + e^{-s} \frac{1}{s} \).

Use this page to start your solution. Attach extra pages as needed.