Name ______________________________ **Lecture Meets at** ______

Differential Equations and Linear Algebra 2250

Midterm Exam 3 [12:25 lecture]
Version 1.12.2010

Scores
3.
4.
5.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

3. **(Chapter 5)** Complete all.

(3a) [40%] Write the solution $x(t)$ of

$$x''(t) + 16x(t) = 196\sin(3t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.
Report the two natural frequencies and the solution $x(t)$.
To save time, don't convert to phase-amplitude form.

Answer:
The frequencies are 4 and 3, the solution is $x(t) = -21\sin(4t) + 28\sin(3t)$.

(3b) [30%] Given $7x''(t) + 12x'(t) + 5x(t) = 0$, which represents a damped spring-mass system with $m = 7$, $c = 12$, $k = 5$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$!

Answer:
Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 12^2 - 4(7)(5) = 4$, therefore there are two distinct roots and the equation is **over-damped**. Alternatively, factor $7r^2 + 12r + 5$ to obtain roots $-5/7$, -1 and then classify as **over-damped**.

(3c) [30%] Consider the variation of parameters formula (33) in Edwards-Penney,

$$y_p(x) = y_1(x)\left(\int \frac{-y_2(x)f(x)}{W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(x)}dx\right).$$

Given the second order equation

$$y''(x) + 8y'(x) + 20y(x) = 5\sin(x^2) + 12\cos(x),$$

write the equations for the variables y_1, y_2, W, f.
To save time, do not write out y_p and do not try to evaluate any integrals.

Answer:
Variables are $y_1(x) = e^{-4x}\cos(2x)$, $y_2(x) = e^{-4x}\sin(2x)$, $f(x) = 5\sin(x^2) + 12\cos(x)$, $W(x) = 2e^{-8x}$.

Use this page to start your solution. Attach extra pages as needed.

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4. (Chapter 5) Complete all.

(4a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 9 with roots $0, 0, 0,\ 3, 3,\ 2i, -2i,\ 2i,\ -2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 3e^x + 4e^{-x} + 5x^3 + 6x\cos 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p.
To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The atom list for $f(x)$ is e^x, e^{-x}, $1, x, x^2, x^3$, $\cos 2x$, $x\cos 2x$, $\sin 2x$, $x\sin 2x$. The list of 10 atoms is broken into 5 groups, each group having exactly one base atom: (1) e^x, (2) e^{-x}, (3) $1, x, x^2, x^3$, (4) $\cos 2x$, $x\cos 2x$ (5) $\sin 2x$, $x\sin 2x$. The modification rule is applied to groups 3 through 5. The trial solution is a linear combination of the replacement 10 atoms in the new list (1) e^x, (2) e^{-x}, (3) x^3, x^4, x^5, x^6, (4) $x^2\cos 2x$, $x^3\cos 2x$ (5) $x^2\sin 2x$, $x^3\sin 2x$. Groups 1, 2 are not modified. The root 3 of the homogeneous equation is not used in the calculation.

(4b) [40%] Let $f(x) = 4xe^x + x\cos 2x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer:

Because $x\cos 2x$ is an atom for the differential equation if and only if $\sin 2x$, $x\sin 2x$, $\cos 2x$ $x\cos 2x$ are atoms, then the characteristic equation must have roots $2i, -2i, 2i, -2i$, listing according to multiplicity. Similarly, xe^x is an atom for the differential equation if and only if 1 is a double root of the characteristic equation. Total of 6 roots: $1, 1, \pm 2i, \pm 2i$ with product of the factors $(r-1)^2(r^2+4)(r^2+4)$ equal to the 6th order characteristic polynomial.

Use this page to start your solution. Attach extra pages as needed.

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5. (Chapter 6) Complete all parts.

(5a) [40%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & -2 & 5 & 6 \\ -4 & 3 & 7 & 8 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$.

To save time, do not find eigenvectors!

Answer:

$-1, 5, 2 \pm 3i$

(5b) [30%] Given $A = \begin{pmatrix} 4 & -1 & 2 \\ 0 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$, which has eigenvalues $3, 3, 2$, find all eigenvectors for eigenvalue 3.

To save time, do not find the eigenvector for eigenvalue 2.

Answer:

One frame sequence is required for $\lambda = 3$. The sequence starts with $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & 1 & -2 \end{pmatrix}$, the last frame having two rows of zeros. There are two invented symbols t_1, t_2 in the last frame algorithm answer $x_1 = t_1 - 2t_2$, $x_2 = t_1$, $x_3 = t_2$. Taking ∂_{t_1} and ∂_{t_2} gives two eigenvectors, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(5c) [40%] Find the matrices P, D in the diagonalization equation $AP = PD$ for the matrix $A = \begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix}$.

Answer:

The eigenpairs are $\left(-1, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$, $\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed.