Name _____ **Lecture Meets at** _____

Differential Equations and Linear Algebra 2250

Midterm Exam 3 [7:30 lecture]
Version 23.11.2010

Scores
1.
2.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$\begin{aligned} x' &= 3x + y, \\ y' &= 4y, \\ x(0) &= 1, \quad y(0) = 1. \end{aligned}$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s-3)\mathcal{L}(x) + (-1)\mathcal{L}(y) = 1, \quad (0)\mathcal{L}(x) + (s-4)\mathcal{L}(y) = 1.$$

Cramer's rule applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{1}{s-4}$. Then the backward table implies $x(t) = e^{4t}$.

(1b) [30%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{5s^2 - 4s + 2}{s^2(s-1)^2}.$$

Answer:

$\mathcal{L}(f(t)) = \frac{2}{s^2} + \frac{3}{(s-1)^2} = \mathcal{L}(2t + 3te^t)$ implies $f(t) = 2t + 3te^t$.

(1c) [30%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) + 2L(tf(t)) = \frac{6}{s^3}.$$

Answer:

Use the s-differentiation theorem and the backward Laplace table to get $(t)f(t) = 3t^2$ or $f(t) = 3t$.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Chapter 10)** Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^{2t}\sin 5t$	
$t^2 e^{t/5}$	
$t\cos 2t$	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{2}{s^2+4}$	$\sin 2t$
$\dfrac{s-2}{s^2-4s+5}$	
$\dfrac{3}{(2s-1)^2}$	
$\dfrac{s}{s^2+6s+10}$	

Answer:

Forward: $\dfrac{5}{(s-2)^2+25}$, $\dfrac{2}{(s-1/5)^3}$, $-\dfrac{d}{ds}\dfrac{s}{s^2+4} = \dfrac{s^2-4}{(s^2+4)^2}$.

Backward: $e^{2t}\cos(t)$, $\dfrac{3}{4}te^{t/2}$, $e^{-3t}\cos t - 3e^{-3t}\sin t$.

(2b) [40%] Find $\mathcal{L}(f(t))$, given $f(t) = e^{-2t}u(t-3)$, where u is the unit step function, $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$.

Answer:

Use the second shifting theorem

$$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t\to t+a}\right).$$

Then $\mathcal{L}\left(e^{-2t}u(t-3))\right) = e^{-3s}\mathcal{L}\left(e^{-2t}\Big|_{t\to t+3}\right) = e^{-3s}\mathcal{L}\left(e^{-2t-6}\right) = e^{-3s-6}\dfrac{1}{s+2}$.

An alternate method, using the first and second shifting theorem, has a few more steps, but it is also correct and easy to visualize. In this case,

$$\mathcal{L}\left(e^{-2t}u(t-3)\right) = \mathcal{L}\left(u(t-3)\right)|_{s\to s+2} = e^{-3s}\mathcal{L}(1)\Big|_{s\to s+2} = \frac{e^{-3s}}{s}\Bigg|_{s\to s+2} = e^{-3s-6}\frac{1}{s+2}.$$

Use this page to start your solution. Attach extra pages as needed.