Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Chapter 10) Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace’s method to solve the system for \( x(t) \). Don’t waste time solving for \( y(t) \! \)

\[
\begin{align*}
x' &= 3x + y, \\
y' &= 4y, \\
x(0) &= 1, & y(0) &= 1.
\end{align*}
\]

Answer:

The Laplace resolvent equation \((sI - A)L(u) = u(0)\) can be written out to find a \(2 \times 2\) linear system for unknowns \(L(x(t)), L(y(t))\):

\[
\begin{align*}
(s - 3)L(x) + (-1)L(y) &= 1, & (0)L(x) + (s - 4)L(y) &= 1.
\end{align*}
\]

Cramer's rule applies to this system to solve for \( L(x(t)) = \frac{1}{s - 4} \). Then the backward table implies \( x(t) = e^{4t} \).

(1b) [30%] Find \( f(t) \) by partial fraction methods, given

\[
L(f(t)) = \frac{5s^2 - 4s + 2}{s^2(s - 1)^2}.
\]

Answer:

\( L(f(t)) = \frac{2}{s^2} + \frac{3}{(s-1)^2} \) implies \( f(t) = 2t + 3te^t \).

(1c) [30%] Solve for \( f(t) \), given

\[
\frac{d}{ds}L(f(t)) + 2L(tf(t)) = \frac{6}{s^3}.
\]

Answer:

Use the \(s\)-differentiation theorem and the backward Laplace table to get \((t)f(t) = 3t^2\) or \(f(t) = 3t\).

Use this page to start your solution. Attach extra pages as needed.
2. (Chapter 10) Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

<table>
<thead>
<tr>
<th>Forward Table</th>
<th>Backward Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$</td>
<td>$\mathcal{L}(f(t))$</td>
</tr>
<tr>
<td>$t^3$</td>
<td>$\frac{6}{s^4}$</td>
</tr>
<tr>
<td>$e^{2t} \sin 5t$</td>
<td></td>
</tr>
<tr>
<td>$t^2e^{t/5}$</td>
<td></td>
</tr>
<tr>
<td>$t \cos 2t$</td>
<td></td>
</tr>
</tbody>
</table>

Answer:
Forward: $\frac{5}{(s - 2)^2 + 25} \cdot \frac{2}{(s - 1/5)^3} - \frac{d}{ds} \frac{s}{s^2 + 4} = \frac{s^2 - 4}{(s^2 + 4)^2}$.
Backward: $e^{2t} \cos(t)$, $\frac{3}{4}t e^{t/2}$, $e^{-3t} \cos t - 3e^{-3t} \sin t$.

(2b) [40%] Find $\mathcal{L}(f(t))$, given $f(t) = e^{-2t}u(t - 3)$, where $u$ is the unit step function, $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$.

Answer:
Use the second shifting theorem

$$\mathcal{L}(g(t)u(t - a)) = e^{-as} \mathcal{L}\left( g(t)|_{t\rightarrow t+a} \right).$$

Then $\mathcal{L}\left( e^{-2t}u(t - 3) \right) = e^{-3s} \mathcal{L}\left( e^{-2t}|_{t\rightarrow t+3} \right) = e^{-3s} \mathcal{L}\left( e^{-2t-6} \right) = e^{-3s-6} \frac{1}{s+2}$.

An alternate method, using the first and second shifting theorem, has a few more steps, but it is also correct and easy to visualize. In this case,

$$\mathcal{L}\left( e^{-2t}u(t - 3) \right) = \mathcal{L}(u(t - 3)|_{s\rightarrow s+2} = e^{-3s} \mathcal{L}(1)|_{s\rightarrow s+2} = \frac{e^{-3s}}{s}|_{s\rightarrow s+2} = e^{-3s-6} \frac{1}{s+2}.$$