Name ________________________________ **Lecture Meets at** ______

Differential Equations and Linear Algebra 2250

Midterm Exam 3 [12:25 lecture]
Version 22.11.2010

Scores
1.
2.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$\begin{aligned} x' &= 2x + y, \\ y' &= 4y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s-2)\mathcal{L}(x) + (-1)\mathcal{L}(y) = 1, \quad (0)\mathcal{L}(x) + (s-4)\mathcal{L}(y) = 2$$

which is a 2×2 system for $x_1 = \mathcal{L}(x)$, $x_2 = \mathcal{L}(y)$:

$$(s-2)x_1 + (-1)x_2 = 1, \quad (0)x_1 + (s-4)x_2 = 2.$$

Cramer's rule applies to this system to solve for $x_1 = \mathcal{L}(x(t)) = \dfrac{1}{s-4}$. Then the backward table implies $x(t) = e^{4t}$. Similarly, $x_2 = \mathcal{L}(y) = \dfrac{2}{s-4}$, and then $y(t) = 2e^{4t}$ ($y(t)$ was not requested).

(1b) [30%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{5s^2 + 4s + 2}{s^2(s+1)^2}.$$

Answer:

$\mathcal{L}(f(t)) = \frac{2}{s^2} + \frac{3}{(s+1)^2} = \mathcal{L}(2t + 3te^{-t})$ implies $f(t) = 2t + 3te^{-t}$.

(1c) [30%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) = L(tf(t)) + \frac{2}{s^3}.$$

Answer:

Use the s-differentiation theorem and the backward Laplace table to get $(-2t)f(t) = t^2$ or $f(t) = -t/2$.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Chapter 10)** Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^{2t}\sin 3t$	
$t^2e^{-t/5}$	
$t\cos t$	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{2}{s^2+4}$	$\sin 2t$
$\dfrac{s+1}{s^2+2s+5}$	
$\dfrac{3}{(2s+1)^2}$	
$\dfrac{s}{s^2-4s+5}$	

Answer:

Forward: $\dfrac{3}{(s-2)^2+9}$, $\dfrac{2}{(s+1/5)^3}$, $-\dfrac{d}{ds}\dfrac{s}{s^2+1} = \dfrac{s^2-1}{(s^2+1)^2}$.

Backward: $e^{-t}\cos(2t)$, $\dfrac{3}{4}te^{-t/2}$, $e^{2t}\cos t + 2e^{2t}\sin t$.

(2b) [40%] Find $\mathcal{L}(f(t))$, given $f(t) = e^{-2t}u(t-\pi)$, where u is the unit step function, $u(t) = 1$ for $t \ge 0$, $u(t) = 0$ for $t < 0$.

Answer:

Use the second shifting theorem

$$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t\to t+a}\right).$$

Then $\mathcal{L}\left(e^{-2t}u(t-\pi)\right) = e^{-\pi s}\mathcal{L}\left(e^{-2t}\Big|_{t\to t+\pi}\right) = e^{-\pi s}\mathcal{L}\left(e^{-2t-2\pi}\right) = e^{-\pi s-2\pi}\dfrac{1}{s+2}$.

An alternate method, using the first and second shifting theorem, has a few more steps, but it is also correct and easy to visualize. In this case,

$\mathcal{L}\left(e^{-2t}u(t-\pi)\right) = \mathcal{L}\left(u(t-\pi)\right)|_{s\to s+2} = e^{-\pi s}\mathcal{L}(1)|_{s\to s+2} = \left.\dfrac{e^{-\pi s}}{s}\right|_{s\to s+2} = e^{-\pi s-2\pi}\dfrac{1}{s+2}$.

Use this page to start your solution. Attach extra pages as needed.