1. (Chapter 10) Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace’s method to solve the system for $x(t)$. Don’t waste time solving for $y(t)$!

$$
x' = 2x + y, \\
y' = 4y, \\
x(0) = 1, \quad y(0) = 2.
$$

Answer:
The Laplace resolvent equation $(sI - A)L(u) = u(0)$ can be written out to find a 2 x 2 linear system for unknowns $L(x(t)), L(y(t))$:

$$(s - 2)L(x) + (-1)L(y) = 1, \quad (0)L(x) + (s - 4)L(y) = 2$$

which is a 2 x 2 system for $x_1 = L(x), x_2 = L(y)$:

$$(s - 2)x_1 + (-1)x_2 = 1, \quad (0)x_1 + (s - 4)x_2 = 2.$$

Cramer’s rule applies to this system to solve for $x_1 = L(x(t)) = \frac{1}{s-4}$. Then the backward table implies $x(t) = e^{4t}$. Similarly, $x_2 = L(y) = \frac{2}{s-4},$ and then $y(t) = 2e^{4t}$ ($y(t)$ was not requested).

(1b) [30%] Find $f(t)$ by partial fraction methods, given

$$L(f(t)) = \frac{5s^2 + 4s + 2}{s^2(s+1)^2}.$$

Answer:
$L(f(t)) = \frac{2}{s^2} + \frac{3}{(s+1)^2} = L(2t + 3te^{-t})$ implies $f(t) = 2t + 3te^{-t}$.

(1c) [30%] Solve for $f(t)$, given

$$\frac{d}{ds}L(f(t)) = L(tf(t)) + \frac{2}{s^3}.$$

Answer:
Use the $s$-differentiation theorem and the backward Laplace table to get $(-2t)f(t) = t^2$ or $f(t) = -t/2$.

Use this page to start your solution. Attach extra pages as needed.
2. (Chapter 10) Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

<table>
<thead>
<tr>
<th>Forward Table</th>
<th>Backward Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$</td>
<td>$\mathcal{L}(f(t))$</td>
</tr>
<tr>
<td>$t^3$</td>
<td>$\frac{6}{s^4}$</td>
</tr>
<tr>
<td>$e^{2t}\sin 3t$</td>
<td></td>
</tr>
<tr>
<td>$t^2e^{-t/5}$</td>
<td></td>
</tr>
<tr>
<td>$t \cos t$</td>
<td></td>
</tr>
</tbody>
</table>

Answer:
Forward: $\frac{3}{(s - 2)^2 + 9}, \frac{2}{(s + 1/5)^3}, -\frac{d}{ds}s^2 + 1 = \frac{s^2 - 1}{(s^2 + 1)^2}$.

Backward: $e^{-t}\cos(2t), \frac{3}{4}te^{-t/2}, e^{2t}\cos t + 2e^{2t}\sin t$.

(2b) [40%] Find $\mathcal{L}(f(t))$, given $f(t) = e^{-2t}u(t - \pi)$, where $u$ is the unit step function, $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$.

Answer:
Use the second shifting theorem

$$\mathcal{L}(g(t)u(t - a)) = e^{-as}\mathcal{L}\left(g(t)|_{t\rightarrow t+a}\right).$$

Then $\mathcal{L}\left(e^{-2t}u(t - \pi)\right) = e^{-\pi s}\mathcal{L}\left(e^{-2t}|_{t\rightarrow t+\pi}\right) = e^{-\pi s}e^{-2\pi} = e^{-\pi s - 2\pi} \frac{1}{s + 2}$.

An alternate method, using the first and second shifting theorem, has a few more steps, but it is also correct and easy to visualize. In this case,

$$\mathcal{L}\left(e^{-2t}u(t - \pi)\right) = \mathcal{L}(u(t - \pi))|_{s\rightarrow s+2} = e^{-\pi s}\mathcal{L}(1)|_{s\rightarrow s+2} = \frac{e^{-\pi s}}{s}|_{s\rightarrow s+2} = e^{-\pi s - 2\pi} \frac{1}{s + 2}.$$