Name KEY Lecture Meets at ______

Scores
1.
2.
3.

Differential Equations and Linear Algebra 2250

Midterm Exam 2 [12:25 lecture]

Version 28.10.2010 Thu

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} b & -1 & 0 \\ -b & 1 & a \\ c-b & 3 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ a^2+a \\ a^2 \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution. $a(2b+c)\neq 0$

(b) [30%] Determine a, b and c such that the system has no solution. $2b+c=0,\ a\neq 0$

(c) [30%] Determine a, b and c such that the system has infinitely many solutions. $a=0$

(a) $\begin{vmatrix} b & -1 & 0 \\ -b & 1 & a \\ c-b & 3 & a \end{vmatrix} = -a(2b+c) \Rightarrow$ unique sol for $a(2b+c)\neq 0$

(b) $\left(\begin{array}{ccc|c} b & -1 & 0 & 0 \\ -b & 1 & a & a^2+a \\ c-b & 3 & a & a^2 \end{array}\right) \xrightarrow{\text{combo}(1,2,1)} \left(\begin{array}{ccc|c} b & -1 & 0 & 0 \\ 0 & 0 & a & a^2+a \\ c-b & 3 & a & a^2 \end{array}\right) \xrightarrow{\text{combo}(1,3,3)} \left(\begin{array}{ccc|c} b & -1 & 0 & 0 \\ 0 & 0 & a & a^2+a \\ 2b+c & 0 & a & a^2 \end{array}\right)$

$2b+c=0$ but $a\neq 0 \Rightarrow$ signal equation $0=a$ from combo(3,2,−1)

(c) $a=0 \Rightarrow$ homogeneous system, with sol $x=y=z=0$, so case (b) does not happen. Then there is a row of zeros in the last frame, which implies ∞-many sols. The values of b,c do not affect the outcome.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: Assume subspace S of $\mathcal{R}^3$ is the span of vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ is in S.

(b) [10%] True or false: The set S of all solutions of the differential equation $y'' + 10y' = 0$ is a subspace of the vector space V of all functions on $-\infty < x < \infty$.

(c) [10%] True or false: Relations $x^2 + y$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

(d) [10%] True or false: Equations $2x + y = 0$, $3y + 4z = 0$ define a subspace in $\mathcal{R}^3$.

(e) [20%] State one theorem, without proof, that concludes that all linear combinations of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ forms a subspace S of the vector space V of all 2×2 matrices.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} -3x_1 & - & 3x_2 & + & 2x_3 & + & 2x_4 & = & 0, \\ -2x_1 & - & 2x_2 & + & 2x_3 & + & 2x_4 & = & 0, \\ x_1 & + & x_2 & & & & & = & 0, \\ -x_1 & - & x_2 & + & x_3 & + & x_4 & = & 0. \end{array}$$

(a) assume that $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \in \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ in other words there exist x_1, x_2

$$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ 0 = 1 \end{cases} \quad \text{contradiction}$$

(b) $y_1'' + 10y_1' = 0$, $y_2'' + 10y_2' = 0$ $\Rightarrow$ $(\alpha_1 y_1 + \alpha_2 y_2)'' + 10[y_1\alpha_1 + \alpha_2 y_2]' = \alpha_1[y_1'' + 10y_1'] + \alpha_2[y_2'' + 10y_2'] =$
$= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$

if $y = 0$ then $y'' + 10y' = 0 + 10 \cdot 0 = 0$

from subspace criterion S is a subspace.

(c) $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in S \quad -1\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \notin S$

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(d) kernel theorem

(e) span theorem

or

subspace criterion

(f)

$$\left[\begin{array}{cccc|c} -3 & -3 & 2 & 2 & 0 \\ -2 & -2 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -t_1$

$x_2 = t_1$

$x_3 = -t_2$

$x_4 = t_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

basis of solutions $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

thu

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3. **(Independence and Dependence)** Do **all** parts.

(a) [10%] State a dependence test for 4 vectors in $\mathcal{R}^4$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State two more dependence tests [different than (a)] for 4 vectors in $\mathcal{R}^4$.

(c) [10%] True or False? The non-pivot columns of a 4×4 matrix A are always linearly dependent.

(d) [30%] Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 5 & 1 \end{pmatrix}.$$

Display the details of an independence test for the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

(e) [40%] Extract from the list below a largest set of independent vectors.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 4 \\ 4 \\ 2 \\ 10 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 5 \end{pmatrix}.$$

(a) If $A = \text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent $\Leftrightarrow$ $\det(A) = 0$. (Determinant test: We can only apply this test because A is square.)

(b) Pivot theorem, rank test.

(c) False: The non-pivot columns of A are always dependent on the pivot columns of A, but the set of non-pivot columns is not necessarily a dependent set by itself.

(d)

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \text{rank}(A) = 3 \Rightarrow \text{L.I. rows.}$$

(e)

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 4 & 0 & 3 & 3 \\ 0 & 2 & 1 & 4 & 0 & 3 & 3 \\ 0 & -2 & -1 & 2 & 2 & -3 & -1 \\ 0 & 2 & 1 & 10 & 2 & 3 & 5 \end{pmatrix}, \quad \text{rref}(B) = \begin{pmatrix} 0 & 1 & 1/2 & 0 & -2/3 & 3/2 & 5/6 \\ 0 & 0 & 0 & 1 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \uparrow \qquad\quad \uparrow$$

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Pivot theorem $\Rightarrow \{v_2, v_4\}$ is a largest L.I. set.

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Differential Equations and Linear Algebra 2250

Midterm Exam 2 [7:30 lecture]
Version Thursday 4.11.2010

Scores
4.
5.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. **(Determinants)** Do all parts.
(a) [20%] State three different rules for $n \times n$ determinants that conclude that the value of the determinant is zero. For example, the first rule says that a zero row or column in A implies $\det(A) = 0$.
(b) [30%] Assume given 4×4 matrices A, B. Suppose $E_3 B^3 = 2(E_2 E_1 A A^T)$ and E_1, E_2, E_3 are elementary matrices representing respectively a combination, a swap and a multiply by 13. Symbol A^T is the transpose of A. Assume $\det(A) = -2$. Find $\det(B)$.

(c) [20%] Determine all values of x for which B^{-1} exists, where $B = \begin{pmatrix} 1 & 4x-1 & 6x+7 & 1 \\ 2 & 0 & 2 & 1 \\ 6x & 0 & 10 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

(d) [30%] Apply the adjugate [adjoint] formula for the inverse $A^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)}$ to find the value of the entry in row 3, column 2 of $(3I + 2B)^{-1}$, where I is the identity matrix and B is given below. Other methods are not acceptable.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{pmatrix}$$

(a) 1) if the matrix A has two same rows or two same columns then the determinant is zero.

2) if one row is a linear combination of other rows then the determinant is zero. Similarily with columns

3) ~~There~~ The matrix is in upper or lower triangular form and there is an element equal to 0 on the diagonal.

4) if the matrix A has a row or column of zeros

5) if rref(A) has a row or column of zeros

(b) E_1 - combination, $|E_1| = 1$

E_2 - swap, $|E_2| = -1$

E_3 - multiply by 13, $|E_3| = 13$

$|A^T| = |A| = -2$

$$E_3 B^3 = 2(E_2 E_1 A A^T)$$

$$|E_3|\,|B|^3 = |2I|\,|E_2|\,|E_1|\,|A|\,|A^T|$$

Use this page to start your solution. Attach extra pages as needed.

$$13\,|B|^3 = 2^4 (-1)\,1\,(-2)(-2)$$

$$|B| = \sqrt[3]{\frac{-2^6}{13}} = \frac{-4}{\sqrt[3]{13}}$$

(c) $\det(B) = 4x(3x+19)$

B^{-1} exists $\Longleftrightarrow$ $|B| \neq 0$ $\Longleftrightarrow$ $x \neq 0$ and $x \neq -\frac{19}{3}$

(d) $A = 3I + 2B = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 7 & 0 \\ -2 & 4 & 1 \end{bmatrix}$

$$A^{-1}[3,2] = \frac{\text{cofactor}(A,2,3)}{|A|} = \frac{(-1)^{2+3} \cdot \begin{vmatrix} 5 & 2 \\ -2 & 4 \end{vmatrix}}{|A|} = \frac{-(20+4)}{75} = \frac{-24}{75} = -\frac{8}{25}$$

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5. (Linear Differential Equations) Do all parts.

(a) [20%] Solve for the general solution of $y''' + 16y'' + 80y' = 0$.

(b) [40%] The characteristic equation is $r^2(r^2+r)^2(r^2+2r+10)^2(r^2+4) = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.

(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions x and $e^{-x}\sin x$. What are the four roots of the characteristic equation?

(d) [20%] A linear homogeneous differential equation with constant coefficients with characteristic equation roots $r = 0, 0, 1, 1, 1, 1, 1+2i, 1-2i$, listed according to multiplicity, is solved incorrectly and then initial conditions are applied to give a particular solution

$$y(x) = xe^x + 2x^3e^x + \frac{5}{3}e^{2x}\sin(x) + 5xe^{-x} + e^{2x}.$$

Circle the terms in the solution which are certainly in error.

(a) $r^3 + 16r^2 + 80r = r((r+8)^2+16) \Rightarrow r = 0, -8 \pm 4i$

atoms $= e^{0x}, e^{-8x}\cos(4x), e^{-8x}\sin(4x)$; $y =$ l.c. of the 3 atoms

(b) $r^4(r+1)^2((r+1)^2+9)^2(r^2+4) = 0$

$r = 0, 0, 0, 0, -1 \pm 3i, -1 \pm 3i, \pm 2i, -1, -1$

atoms $= 1, x, x^2, x^3$

e^{-x}, xe^{-x}

$e^{-x}\cos 3x, xe^{-x}\cos 3x$

$e^{-x}\sin 3x, xe^{-x}\sin 3x$

$\cos 2x,$

$\sin 2x$

$y =$ l.c. of the 12 atoms

(c) $x \Rightarrow$ root $0 \Rightarrow$ atoms $1, x$ and roots $= 0, 0$

$e^{-x}\sin x \Rightarrow$ roots $-1 \pm i \Rightarrow$ atoms $e^{-x}\cos x, e^{-x}\sin x$ and roots $-1 \pm i$

$\boxed{\text{Roots} = 0, 0, -1 \pm i}$

(d) Last three have roots $2 \pm i, -1, -1, 2$. None of these are roots of the characteristic equation. The first two terms are sols of the homogeneous DE, because their roots would be $1, 1, 1, 1$, which matches the characteristic equation roots.

Use this page to start your solution. Attach extra pages as needed.