Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (The 3 Possibilities with Symbols)
   Let \( a, b \) and \( c \) denote constants and consider the system of equations
   \[
   \begin{pmatrix}
   b & -1 & 0 \\
   -b & 1 & a \\
   c-b & 3 & a
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y \\
   z
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 \\
   a^2 + a \\
   a^2
   \end{pmatrix}
   \]

   (a) [40%] Determine \( a, b \) and \( c \) such that the system has a unique solution.
   \[ a(2b+c) \neq 0 \]

   (b) [30%] Determine \( a, b \) and \( c \) such that the system has no solution.
   \[ 2b+c = 0, \ a \neq 0 \]

   (c) [30%] Determine \( a, b \) and \( c \) such that the system has infinitely many solutions.
   \[ a = 0 \]

(a) \[
\begin{vmatrix}
-1 & a \\
-1 & a \\
-c & 3 & a
\end{vmatrix}
= -a(2b+c) \Rightarrow \text{unique sol} \ \text{if} \ a(2b+c) \neq 0
\]

(b) \[
\begin{pmatrix}
b & -1 & 0 \\
-l & 1 & a \\
-c-b & 3 & a
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & a^2+1 \\
0 & 0 & a^2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
b & -1 & 0 \\
0 & 0 & a^2 \\
2b+c & 0 & a
\end{pmatrix}
\]
   Combo(1, 2, 1)

   \[ 2b+c = 0 \text{ but } a \neq 0 \Rightarrow \text{signature equation } 0 = a \text{ from combo(3, 2, -1)} \]

(c) \[ a = 0 \Rightarrow \text{homogeneous system with sol } x = y = z = 0, \text{ so case (b) does not happen. Then there is a row of zeros in the last frame, which implies } \infty \text{ - many sols. The values of } b, c \text{ do not affect the outcome.} \]

Use this page to start your solution. Attach extra pages as needed.
2. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or [False]: Assume subspace $S$ of $\mathbb{R}^3$ is the span of vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ is in $S$.

(b) [10%] True or False: The set $S$ of all solutions of the differential equation $y'' + 10y' = 0$ is a subspace of the vector space $V$ of all functions on $-\infty < x < \infty$.

(c) [10%] True or False: Relations $x^2 + y, y + z = 0$ define a subspace in $\mathbb{R}^3$.

(d) [10%] True or False: Equations $2x + y = 0, 3y + 4z = 0$ define a subspace in $\mathbb{R}^3$.

(e) [20%] State one theorem, without proof, that concludes that all linear combinations of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ forms a subspace $S$ of the vector space $V$ of all $2 \times 2$ matrices.

(f) [40%] Find a basis of vectors for the subspace of $\mathbb{R}^4$ given by the system of restriction equations

$$
\begin{align*}
-3x_1 - 3x_2 + 2x_3 + 2x_4 &= 0, \\
-2x_1 - 2x_2 + 2x_3 + 2x_4 &= 0, \\
x_1 + x_2 &= 0, \\
-x_1 - x_2 + x_3 + x_4 &= 0.
\end{align*}
$$

(c) Assume that $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ in other words there exist $x, x_2$

$$
\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}
$$

\[ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = 0 \text{ contradiction.}
\]

(e) $y_1'' + 10y_1' = 0 \Rightarrow (\alpha_1 y_1 + \alpha_2 y_2)^{''} + 10(\alpha_1 y_1 + \alpha_2 y_2)' = \alpha_1 (y_1^{''} + 10y_1') + \alpha_2 (y_2^{''} + 10y_2') = 0$.

If $y_2 \in S$ then $y_2^{''} + 10y_2' = 0 + 10 \cdot 0 = 0$.

From subspace criterion $S$ is a subspace.

(e) $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \in S, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \notin S$. 

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(d) kernel theorem
(e) span theorem

subspace criterion

\[
\begin{pmatrix}
-3 & -3 & 2 & 2 & 0 \\
-2 & -2 & 2 & 2 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
x_1 = -t_1 \\
x_2 = t_1 \\
x_3 = -t_2 \\
x_4 = t_2 
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = t_1 \begin{bmatrix}
-1 \\
0 \\
0 \\
0
\end{bmatrix} + t_2 \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

basis of solutions:

\[
\begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}
\]
3. (Independence and Dependence) Do all parts.
   (a) [10%] State a dependence test for 4 vectors in \( \mathbb{R}^4 \). Write the hypothesis and conclusion, not just the name of the test.
   (b) [10%] State two more dependence tests [different than (a)] for 4 vectors in \( \mathbb{R}^4 \).
   (c) [10%] True or False? The non-pivot columns of a \( 4 \times 4 \) matrix \( A \) are always linearly dependent.
   (d) [30%] Let \( u_1, u_2, u_3 \) denote the rows of the matrix
   \[
   A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 5 & 1 \end{pmatrix}.
   \]
   Display the details of an independence test for the vectors \( u_1, u_2, u_3 \).
   (e) [40%] Extract from the list below a largest set of independent vectors.
   \[
   \begin{align*}
   v_1 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 10 \end{pmatrix}, \\
   v_5 &= \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 3 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \quad v_7 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 5 \end{pmatrix}.
   \end{align*}
   \]
   (a) If \( A = \text{span}(v_1, v_2, v_3, v_4) \), then \( v_1, v_2, v_3, v_4 \) are dependent \( \iff \det(A) = 0 \). 
   (Determinant test: We can only apply this test because \( A \) is square.)
   (b) Pivot theorem, rank test.
   (c) False: The non-pivot columns of \( A \) are always dependent on the pivot columns of \( A \), but the set of non-pivot columns is not necessarily a dependent set by itself.
   (d) 
   \[
   \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \text{rank}(A) = 3 \Rightarrow \text{L.I. rows}.
   \]
   (e) 
   \[
   B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 4 & 0 \\ 0 & 2 & 1 & 4 & 0 \\ 0 & -2 & -1 & 2 & -3 \\ 0 & 2 & 1 & 10 & 2 & -3 \end{pmatrix}, \quad \text{rref}(B) = \begin{pmatrix} 0 & 1 & 1/2 & 0 & -2/3 & 3/2 & 5/6 \\ 0 & 0 & 0 & 1 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
   \]
   \[
   \text{pivot theorem } \Rightarrow \{v_2, v_4\} \text{ is a largest L.I. set}.
   \]
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4. (Determinants) Do all parts.
   (a) [20%] State three different rules for \( n \times n \) determinants that conclude that the value of the determinant is zero. For example, the first rule says that a zero row or column in \( A \) implies \( \det(A) = 0 \).
   (b) [30%] Assume given \( 4 \times 4 \) matrices \( A, B \). Suppose \( E_3 B^3 = 2(E_2 E_1 A A^T) \) and \( E_1, E_2, E_3 \) are elementary matrices representing respectively a combination, a swap and a multiply by 13. Symbol \( A^T \) is the transpose of \( A \). Assume \( \det(A) = -2 \). Find \( \det(B) \).
   (c) [20%] Determine all values of \( x \) for which \( B^{-1} \) exists, where \( B = \begin{pmatrix} 1 & 4x - 1 & 6x + 7 & 1 \\ 2 & 0 & 2 & 1 \\ 6x & 0 & 10 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \).
   (d) [30%] Apply the adjugate formula for the inverse \( A^{-1} = \frac{\text{adj}(A)}{\det(A)} \) to find the value of the entry in row 3, column 2 of \( (3I + 2B)^{-1} \), where \( I \) is the identity matrix and \( B \) is given below. Other methods are not acceptable. 

\[
B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{pmatrix}
\]

2) if the matrix \( A \) has two same rows or two same columns then the determinant is zero.
3) if one row is a linear combination of other rows then the determinant is zero. Similarly with columns.
4) The matrix is in upper or lower triangular form and there is an element equal to 0 on the diagonal.
5) if the matrix \( A \) has a row or column of zeros

(b) \( E_1 \) - combination, \( |E_1| = 1 \)
   \( E_2 \) - swap, \( |E_2| = -1 \)
   \( E_3 \) - multiply by 13, \( |E_3| = 13 \)

\[
|A^T| = |A| = -2
\]

\[
E_2 B^3 = 2^3 (E_2 E_1 A A^T)
\]

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(c) \( \det(B) = 4 \times (3x + 18) \)

\[ B^{-1} \text{ exists} \iff |B| \neq 0 \iff x \neq 0 \text{ and } x \neq -\frac{18}{3} \]

(d) \( A = 3I + 2B = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 7 & 0 \\ -2 & 4 & 1 \end{bmatrix} \)

\[ A^{-1}[3,2] = \frac{\text{cofactor}(A_{1,2,3})}{|A|} = \frac{(-1)^{2+3} \begin{vmatrix} 5 & 2 \\ -2 & 4 \end{vmatrix}}{|A|} = \frac{-(20 + 4)}{75} = -\frac{24}{75} = -\frac{8}{25} \]
5. (Linear Differential Equations) Do all parts.
   (a) [20%] Solve for the general solution of \( y'''' + 16y'' + 80y' = 0 \).
   (b) [40%] The characteristic equation is \( r^2(r^2 + r)^2(r^2 + 2r + 10)^2(r^2 + 4) = 0 \). Find the general solution \( y \) of the linear homogeneous constant-coefficient differential equation.
   (c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions \( x \) and \( e^{-x} \sin x \). What are the four roots of the characteristic equation?
   (d) [20%] A linear homogeneous differential equation with constant coefficients with characteristic equation \( r = 0, 0, 1, 1, 1, 1 + 2i, 1 - 2i \), listed according to multiplicity, is solved incorrectly and then initial conditions are applied to give a particular solution
   \[
y(x) = x e^x + 2x^3 e^x + \frac{5}{3} e^{2x} \sin x + 5xe^{-x} + e^{2x}.
   \]
   Circle the terms in the solution which are certainly in error.

- (a) \( r^3 + 16r^2 + 80r = r((r+8)^2 + 1) \Rightarrow r = 0, -8, 4i \)
  atoms = \( e^0x, e^{-8x} \cos(4x), e^{-8x} \sin(4x) \).
  \( y = 1 \text{.c. of } 3 \text{ atoms} \)

- (b) \( r^4((r+1)^2 + 9)^2(r^2 + 4) = 0 \)
  \( r = 0, 0, 0, 0, -1, 3i, -1, -3i, -2i, -1, -1, -1 \)
  atoms = \( \{1, x, x^2, x^3 \}
  \( e^x, e^{-x} \cos(2x), e^{-x} \sin(2x), e^x \cos(3x), e^{-x} \sin(3x), \cos(2x), \sin(2x) \)
  \( y = 1 \text{.c. of } 12 \text{ atoms} \)

- (c) \( x \Rightarrow \text{ not } 0 \Rightarrow \text{ atoms } 1, x \text{ and roots } 0, 0 \)
  \( e^x \sin x \Rightarrow \text{ roots } -1, -1 \Rightarrow \text{ atoms } e^x \cos x, e^x \sin x \text{ and roots } -1, \pm i \)
  \( \text{ Roots are } 0, 0, -1, \pm i \)

- (d) Last three have roots 2i, -1, -1, 2. None of these are roots of the homogeneous DE, because their roots would be 1, 1, 1, 1, which matches the characteristic equation roots.

Use this page to start your solution. Attach extra pages as needed.