Name KEY Lecture Meets at ______

Scores
1.
2.
3.

Differential Equations and Linear Algebra 2250

Midterm Exam 2 [12:25 lecture]
Version 27.10.2010 Wed

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} -b & -1 & 0 \\ b & 1 & a \\ c+b & 3 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ a^2+a \\ a^2 \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution. $a(-2b+c) \neq 0$

(b) [30%] Determine a, b and c such that the system has no solution. $-2b+c=0,\quad a\neq 0$

(c) [30%] Determine a, b and c such that the system has infinitely many solutions. $a=0$

(a) $\begin{vmatrix} -b & -1 & 0 \\ b & 1 & a \\ c+b & 3 & a \end{vmatrix} = -a(-2b+c) \Rightarrow$ unique sol for $a(-2b+c)\neq 0$

(b) $\left(\begin{array}{ccc|c} -b & -1 & 0 & 0 \\ b & 1 & a & a^2+a \\ c+b & 3 & a & a^2 \end{array}\right) \xrightarrow{\text{combo}(1,2,1)} \left(\begin{array}{ccc|c} -b & -1 & 0 & 0 \\ 0 & 0 & a & a^2+a \\ c+b & 3 & a & a^2 \end{array}\right) \xrightarrow{\text{combo}(1,3,3)} \left(\begin{array}{ccc|c} -b & -1 & 0 & 0 \\ 0 & 0 & a & a^2+a \\ c-2b & 0 & a & a^2 \end{array}\right)$

When $c-2b=0$ (so that (a) fails), then there is a signal equation $0=a$ (from combo$(3,2,-1)$) for any $a\neq 0$. No sol for $a\neq 0$ and $c-2b=0$. Part (a) fails also for $a=0$, but this is handled in (c) below.

(c) When $a=0$, then the system is homogeneous with at least the sol $x=y=z=0$, so case (b) does not happen. There is a row of zeros in the last frame of (b). This implies ∞-many sols for $a\neq 0$. There is no condition on symbols b,c.

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Name. KEY

2. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: Assume subspace S of $\mathcal{R}^3$ contains vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$. Then $\begin{pmatrix}3\\2\\1\end{pmatrix}$ is in S.

(b) [10%] True or false: The set of all linear combinations of functions x, e^x, $\sin(x)$ forms a subspace S of the vector space of all functions on $-\infty < x < \infty$.

(c) [10%] True or false: Relations $\ln(1+x^2) = 0$, $y+z=0$ define a subspace in $\mathcal{R}^3$.

(d) [10%] True or false: Equations $2x+y=0$, $3y+4z=0$ define a subspace in $\mathcal{R}^3$.

(e) [20%] State one theorem, without proof, that concludes that all linear combinations of the matrices $\begin{pmatrix}1 & 0\\2 & 2\end{pmatrix}$ and $\begin{pmatrix}2 & 0\\1 & 3\end{pmatrix}$ forms a subspace S of the vector space V of all 2×2 matrices.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl}
-3x_1 & + & 0x_2 & + & 2x_3 & + & 2x_4 & = & 0,\\
-2x_1 & & & + & 2x_3 & + & 2x_4 & = & 0,\\
x_1 & & & & & & & = & 0,\\
-x_1 & & & + & x_3 & + & x_4 & = & 0.
\end{array}$$

(a) $2\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$

(b) span theorem

(c) $\begin{cases}\ln(1+x^2)=0\\ y+z=0\end{cases} \iff \begin{cases}x=0\\ y+z=0\end{cases}$ kernel theorem

(d) $\begin{cases}2x+y=0\\ 3y+4z=0\end{cases}$ kernel theorem

(e) Span theorem

Let $v_1, v_2, \ldots, v_k$ be vectors in the space V. Then the set $\text{span}(v_1, \ldots, v_k)$ of all linear combinations of $v_1, v_2, \ldots, v_k$ is a subspace of V.

V - space of matrices 2×2.

$k=2$

$v_1 = \begin{pmatrix}1 & 0\\2 & 2\end{pmatrix}$

$v_2 = \begin{pmatrix}2 & 0\\1 & 3\end{pmatrix}$

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$$\left[\begin{array}{cccc|c} -3 & 0 & 2 & 2 & 0 \\ -2 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$

$x_2 = t_1$

$x_3 = -t_2$

$x_4 = t_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ t_1 \\ -t_2 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

basis of solutions: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Name. KEY

3. **(Independence and Dependence)** Do **all** parts.

(a) [10%] State an independence test for 3 vectors in $\mathcal{R}^3$. Write the hypothesis and conclusion, not just the name of the test.

(b) [10%] State two more independence tests [different than (a)] for 3 vectors in $\mathcal{R}^3$.

(c) [10%] **Define** what it means for column 2 of a 4×4 matrix A to be a non-pivot column of A.

(d) [30%] Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 5 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{pmatrix}.$$

Display the details of an independence test for the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

(e) [40%] Extract from the list below a largest set of independent vectors.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 4 \\ 4 \\ 2 \\ 10 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 5 \end{pmatrix}.$$

(a) If $A = \text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent $\Leftrightarrow$ $\det(A) \neq 0$. (Determinant test) * We can only apply this test since A is square.

(b) rank test, pivot theorem

(c) Col. 2 of a 4x4 matrix A is a non-pivot column of A if it can be written as a linear combination of the pivot columns, which are the columns of A which correspond to columns of rref(A) with leading ones.

(d)

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \text{rank}(A) = 3 \Rightarrow \text{L.I. rows.}$$

(e)

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 4 & 0 & 3 \\ 0 & 2 & 1 & 3 & 4 & 0 & 3 \\ 0 & -2 & -1 & -3 & 2 & 2 & -1 \\ 0 & 2 & 1 & 3 & 10 & 2 & 5 \end{pmatrix}, \quad \text{rref}(B) = \begin{pmatrix} 0 & 1 & 1/2 & 3/2 & 0 & -2/3 & 5/6 \\ 0 & 0 & 0 & 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Pivot theorem $\Rightarrow \{v_2, v_5\}$ is a largest L.I. set.

Name *KEY* Lecture Meets at ______

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Version Wednesday 3.11.2010

Scores
4.
5.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. **(Determinants)** Do all parts.

(a) [20%] Cramer's Rule for the system $A\mathbf{x} = \mathbf{v}$ displays the unique solution $\mathbf{x}$ componentwise as quotients of $n \times n$ determinants. Write out, without computing any determinant values, the formulas for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) [30%] Assume given 4×4 matrices A, B. Suppose $E_3 B^3 = 2(E_2 E_1 A)$ and E_1, E_2, E_3 are elementary matrices representing respectively a combination, a swap and a multiply by 5. Assume $\det(A) = -4$. Find $\det(B)$.

(c) [20%] Determine all values of x for which B^{-1} exists, where $B = \begin{pmatrix} 1 & 2x-1 & 3x+7 & 1 \\ 2 & 0 & 2 & 1 \\ 3x & 0 & 10 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

(d) [30%] Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 2 of $(2I + B)^{-1}$, where I is the identity matrix and B is given below. Other methods are not acceptable.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{pmatrix}$$

(a) $x_1 = \dfrac{\begin{vmatrix} v_1 & b \\ v_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $x_2 = \dfrac{\begin{vmatrix} a & v_1 \\ c & v_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$

(b) $|E_3||B||B||B| = |2I||E_2||E_1||A| \Rightarrow 5|B|^3 = 2^4(-1)(1)(-4) \Rightarrow |B| = \sqrt[3]{\frac{64}{5}}$

observe I is $4\times 4 \Rightarrow |2I| =$ product of diagonal elements $= 2\cdot 2\cdot 2\cdot 2 = 2^4$

(c) B^{-1} exists $\Leftrightarrow |B| \neq 0$. Expand $|B|$ by cofactors along col $= 4$, Then

$|B| = (-1)(1)\begin{vmatrix} 2 & 0 & 2 \\ 3x & 0 & 10 \\ 1 & 1 & 1 \end{vmatrix} + (1)(1)\begin{vmatrix} 1 & 2x-1 & 3x+7 \\ 3x & 0 & 10 \\ 1 & 1 & 1 \end{vmatrix} = (-1)(-1)(20-6x) + (3x^2+44x-20)$

$|B| = 3x^2 + 38x$ $\quad \boxed{x \neq 0 \text{ and } x \neq -38/3}$

(d) Entry row 3, col 2 $= \dfrac{\mathrm{Cof}(A,2,3)}{|A|}$ where $A = 2I + B = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 0 \\ -1 & 2 & 1 \end{pmatrix}$

$= \dfrac{(-1)\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}}{|A|}$

$= \dfrac{-7}{17}$

Use this page to start your solution. Attach extra pages as needed.

Name. KEY

5. (Linear Differential Equations) Do all parts.

(a) [20%] Solve for the general solution of $y''' + 8y'' + 32y' = 0$.

(b) [40%] The characteristic equation is $r^2(r^2 - r)^2(r^2 + 2r + 5)^2(r^2 + 16) = 0$. Find the general solution y of the linear homogeneous constant-coefficient differential equation.

(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has two solutions $e^{2x}\cos x$ and $e^{-x}\sin x$. What are the roots of the characteristic equation?

(d) [20%] A linear homogeneous differential equation with constant coefficients with characteristic equation roots $r = 0, 0, 1, 1, 1, 1 + 2i, 1 - 2i$, listed according to multiplicity, is solved incorrectly and then initial conditions are applied to give a particular solution

$$y(x) = xe^x + 2x^3e^x + \frac{11}{7}e^x \sin(2x) + 15xe^{-x} + e^{-x}.$$

Circle the terms in the solution which are certainly in error.

(a) $r((r+4)^2+16)=0$, roots $= 0, -4 \pm 4i$, atoms $= 1, e^{-4x}\cos(4x), e^{-4x}\sin(4x)$

$y =$ l.c. of 3 atoms

(b) $r^4(r-1)^2((r+1)^2+4)^2(r^2+16)=0$

$r = 0,0,0,0,1,1,-1\pm 2i, -1\pm 2i, \pm 4i$

atoms $= 1, x, x^2, x^3$

e^x, xe^x

$e^{-x}\cos(2x), xe^{-x}\cos(2x)$

$e^{-x}\sin(2x), xe^{-x}\sin(2x)$

$\cos(4x)$

$\sin(4x)$

$y =$ l.c. of the 12 atoms

(c) $e^{ax}\cos(bx)$, $e^{ax}\sin(bx)$ come from Euler's Theorem with roots $a \pm ib$

$\boxed{2 \pm i, \ -1 \pm i}$

(d) roots $= 1,1,1,1$ produce x^3e^x, by Euler's Theorem

roots $= -1,-1$ produce xe^{-x}, "

root $= -1$ produces e^{-x},

Because -1 is not a root of the characteristic equation, then the last two terms (involving e^{-x}) are in error.

The term $2x^3e^x$ is an error because 1 is a triple root of the characteristic equation, not a quad root.

Use this page to start your solution. Attach extra pages as needed.