Name KEY

Lecture Meets at ______

Scores
1.
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Differential Equations and Linear Algebra 2250

Midterm Exam 2 [12:25 lecture]

Version 25.10.2010 MON

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(The 3 Possibilities with Symbols)**
   Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} b-c & -1 & 0 \\ -b+c & 1 & a \\ 2b+4c & 3 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a^2-2a \\ a^2-a \end{pmatrix}$$

(a) [40%] Determine a, b and c such that the system has a unique solution. $a(5b+c) \neq 0$

(b) [30%] Determine a, b and c such that the system has no solution. $5b+c=0$, $a \neq 0$

(c) [30%] Determine a, b and c such that the system has infinitely many solutions. $a=0$

(a) $\begin{vmatrix} b-c & -1 & 0 \\ -b+c & 1 & a \\ 2b+4c & 3 & a \end{vmatrix} = -a(5b+c) \Rightarrow$ unique sol. when $a(5b+c)\neq 0$

(b) $\left(\begin{array}{ccc|c} b-c & -1 & 0 & a \\ -b+c & 1 & a & a^2-2a \\ 2b+4c & 3 & a & a^2-a \end{array}\right) \xrightarrow{\text{combo}(1,2,1)} \left(\begin{array}{ccc|c} b-c & -1 & 0 & a \\ 0 & 0 & a & a^2-a \\ 2b+4c & 3 & a & a^2-a \end{array}\right) \xrightarrow{\text{combo}(1,3,3)} \left(\begin{array}{ccc|c} b-c & -1 & 0 & a \\ 0 & 0 & a & a^2-a \\ 5b+c & 0 & a & a^2+2a \end{array}\right)$

No sol. when $5b+c=0$, $a\neq 0$ because of a signal equation, namely $0=3a$ from combo$(2,3,-1)$

(c) $a=0$ gives a homogeneous system of equations with $x=y=z=0$ as a solution. So this is not case (a) or (b). There is a row of zeros in the last frame of (b) $\Rightarrow$ ∞-many sols for $a=0$.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Vector Spaces)** Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is a subspace S of $\mathcal{R}^3$ which does not contain the vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(b) [10%] True or false: The set of solutions $\mathbf{x}$ in $\mathcal{R}^3$ of a matrix equation $A\mathbf{x} = \mathbf{0}$ cannot equal all vectors in $\mathcal{R}^3$.

(c) [10%] True or false: Relations $|x + y| = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

(d) [10%] True or false: Equations $x + y = 0$, $y + z = 0$ define a subspace in $\mathcal{R}^3$.

(e) [20%] State one theorem, without proof, that concludes that all linear combinations of the functions x, $\cos x$, e^{x^2} forms a vector space of functions.

(f) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} -3x_1 & + & 10x_2 & + & 2x_3 & + & 4x_4 & = & 0, \\ -2x_1 & + & 4x_2 & + & x_3 & + & 2x_4 & = & 0, \\ 2x_1 & + & 4x_2 & & & & & = & 0, \\ -2x_1 & + & 12x_2 & + & 2x_3 & + & 4x_4 & = & 0. \end{array}$$

(a) $\{ (x, y, z) : x = 0 \}$

(b) if $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $Ax = 0$ is satisfied for all x

(c) $\begin{cases} |x+y| = 0 \\ y+z = 0 \end{cases} \iff \begin{cases} x+y = 0 \\ y+z = 0 \end{cases}$ kernel theorem

(d) kernel theorem

(e) span theorem

or

subspace criterion

$$\left[\begin{array}{cccc|c} -3 & 10 & 2 & 4 & 0 \\ -2 & 4 & 1 & 2 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ -2 & 12 & 2 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}s - \frac{1}{2}t \\ -\frac{1}{8}s - \frac{1}{4}t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

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basis of solutions $\left\{ \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix} \right\}$

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3. **(Independence and Dependence)** Do **all parts.**
(a) [10%] State an independence test for 3 vectors in $\mathcal{R}^5$. Write the hypothesis and conclusion, not just the name of the test.
(b) [10%] State another [different than (a)] independence test for 3 vectors in $\mathcal{R}^5$.
(c) [10%] **Define** what it means for column 2 of a 10×10 matrix A to be a pivot column of A.
(d) [30%] Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_2$, $\mathbf{v}_4$ denote the rows of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 5 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & -6 & 0 \end{pmatrix}.$$

Display the details of a dependence test for the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$.
(e) [40%] Extract from the list below a largest set of independent vectors.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 9 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 5 \end{pmatrix}.$$

(a) Vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ in R^5 are independent if and only if their augmented matrix has rank 3.

(b) Pivot Theorem.

(c) Column 2 of the 10×10 matrix A is a pivot column of A provided that rref(A) has a leading one in column 2.

(d)
$$\text{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(A) = 2 \Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \text{ form a dependent set.}$$

(e)
$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 & 3 \\ 0 & 2 & 1 & 3 & 0 & 3 \\ 0 & -2 & -1 & 3 & 2 & -1 \\ 0 & 2 & 1 & 9 & 2 & 5 \end{pmatrix} \text{ has rref}(B) = \begin{pmatrix} 0 & 1 & 1/2 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & 1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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so by the Pivot Theorem, $\{v_2, v_4\}$ is a largest independent set.

Name $\underline{\text{KEY}}$ Lecture Meets at ______

Differential Equations and Linear Algebra 2250

Midterm Exam 2 [12:25 lecture]
Version Monday 1.11.2010

Scores
4.
5.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. **(Determinants)** Do all parts.

(a) [10%] The adjugate formula is $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = \det(A)I$. Write out this formula in detail for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) [30%] Assume given 4×4 matrices A, B. Suppose $E_3B = 2(E_2E_1A^2)$ and E_1, E_2, E_3 are elementary matrices representing respectively a combination, a multiply by 3 and a swap. Assume $\det(A) = 5$. Find $\det(B)$.

(c) [30%] Determine all values of x for which B^{-1} fails to exist, where $B = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 3x & 0 & 10 \\ 1 & 1 & 2x-1 & 3x+7 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

(d) [30%] Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 2, column 3 of $(I+B)^{-1}$, where I is the identity matrix and B is given below. Other methods are not acceptable.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{pmatrix}$$

(b) $|E_3||B| = |2I||E_2||E_1||A||A| \Rightarrow (-1)|B| = 2^4(3)(1)(5)(5)$
$\Rightarrow |B| = -(16)(75)$

(c) $|B| = (2x-1)(+1)\,\text{minor}(B,3,3) + (1)(-1)\,\text{minor}(B,4,3) = -x(3x+38)$

B^{-1} fails to exist $\Leftrightarrow |B| = 0$
$\Leftrightarrow x = 0$ or $x = \frac{-38}{3}$

(d) $A = I + B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ -1 & 2 & 0 \end{pmatrix} \Rightarrow |A| = (1)(+1)\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 5$

$$\text{Entry } 2,3 = \frac{\text{cofactor}(A,3,2)}{|A|} = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}(-1)^{3+2}}{5} = \frac{1}{5}$$

Use this page to start your solution. Attach extra pages as needed.

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5. **(Linear Differential Equations)** Do all parts.
(a) [20%] Solve for the general solution of $y''' + 8y'' + 20y' = 0$.
(b) [40%] The roots of the characteristic equation are $r = 0, 0, 0, 2+i, 2-i, -1, -1, 2+3i, 2-3i, 2+3i, 2-3i, i, -i$, listed according to multiplicity. Find the general solution y of the linear homogeneous constant-coefficient differential equation.
(c) [20%] A third order linear homogeneous differential equation with constant coefficients has two solutions $e^{2x}\cos x$ and e^{-x}. What are the roots of the characteristic equation?
(d) [20%] A linear homogeneous differential equation with constant coefficients with characteristic equation $(r-1)^2(r^2-1)^2(r^2+2r+17)^2 = 0$ is solved incorrectly and then initial conditions are applied to give a particular solution

$$y(x) = \boxed{x^2e^{-x} + e^x\cos(4x)} + e^x(1+2x).$$ incorrect Terms

Circle the terms in the solution which are certainly in error.

(a) $r^3 + 8r^2 + 20r = 0$
$r(r^2+8r+20) = 0 \Rightarrow r((r+4)^2+4) = 0 \Rightarrow r = 0, -4 \pm 2i$
atoms $= 1, e^{-4x}\cos 2x, e^{-4x}\sin 2x$; y = linear combination of the atoms

(b) atoms $= 1, x, x^2, e^{2x}\cos x, e^{2x}\sin x, e^{-x}, xe^{-x}, e^{2x}\cos 3x, e^{2x}\sin 3x,$
$xe^{2x}\cos 3x, xe^{2x}\sin 3x, \cos x, \sin x$
y = linear combination of the 13 atoms

(c) $e^{2x}\cos x \to 2+i$; roots $= 2+i, 2-i$
$e^{-x} \to -1$; root $= -1$

$\boxed{\text{roots} = -1, 2+i, 2-i}$

(d) $(r-1)^4(r+1)^2((r+1)^2+16)^2 = 0$
roots $= 1, 1, 1, 1, -1, -1, -1+4i, -1-4i, -1+4i, -1-4i$ (10 roots)
atoms $= e^x, xe^x, x^2e^x, x^3e^x,$
$e^{-x}, xe^{-x},$
$e^{-x}\cos 4x, e^{-x}\sin 4x, xe^{-x}\cos 4x, xe^{-x}\sin 4x$

Use this page to start your solution. Attach extra pages as needed.