Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [12:25 lecture]
Version 22.9.2010

Scores
1.
2.
3.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \dfrac{1+x^3}{1+x^2}$.

   (b) [25%] Solve $y' = (\cos x + 1)(\sin x - 1)$.

   (c) [25%] Solve $y' = e^x \ln|1+e^x|$, $y(0) = 1$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(10e^{4t}v(t)\right) = -32e^{4t}$, $v(0) = -\frac{4}{5}$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 0$.

1(a) $y' = x + \dfrac{1}{1+x^2} - \dfrac{x}{1+x^2}$ by the division algorithm

$$\boxed{y = \frac{x^2}{2} + \tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + c}$$

1(b) $y' = \cos x \sin x - \cos x + \sin x - 1$

$$\boxed{y = \frac{\sin^2 x}{2} - \sin x - \cos x - x + c}$$

1(c) $y' = \ln|u|\,du$ where $u = 1+e^x$, $du = e^x dx$

$y = u\ln|u| - u + c$ $\quad y(0)=1 \Rightarrow 1 = 2\ln 2 - 2 + c$

$$\boxed{y = (1+e^x)\ln(1+e^x) - (1+e^x) + 3 - 2\ln 2}$$

1(d) $\int \frac{d}{dt}(10e^{4t}v)dt = -32\int e^{4t}dt \Rightarrow 10e^{4t}v = -8e^{4t} + c$

$v(0) = \frac{-4}{5} \Rightarrow 10(1)(\frac{-4}{5}) = -8(1) + c \Rightarrow c = 0$

$$\boxed{v(t) = \frac{-4}{5}}$$

$x(t) = \int v(t)dt = -\frac{4t}{5} + c_1$

$x(0) = 0 \Rightarrow c_1 = 0$

$$\boxed{x(t) = -\frac{4t}{5}}$$

Use this page to start your solution. Attach extra pages as needed. maple checks (a)→(d); ✓

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

X	$y' + xy = y(y + x) + 2xy^2$	X	$y' = (x-1)(y+1) - (1-x)y$
X	$y' = e^{x+y} - 3e^x e^y$		$x^3 y' = xy^2 + x^2 y$

(b) [10%] State a partial derivative test that decides if $y' = f(x, y)$ is a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = x^3 + y$ is linear and not quadrature. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + \sin(y)$ is not separable. Supply all details.

(b) $y' = f(x,y)$ Quadrature $\Longleftrightarrow$ $\frac{\partial f}{\partial y} = 0$

(c) Linear $\Longleftrightarrow$ $\frac{\partial f}{\partial y}$ is independent of y.

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + y) = 1$ independent of y $\Longrightarrow$ $\boxed{\text{Linear}}$

$\frac{\partial f}{\partial y} = 1 \neq 0$ $\Longrightarrow$ $\boxed{\text{Not quadrature}}$ used test (b)

(d) $y' = f(x,y)$ is not separable if $\frac{\partial f/\partial x}{f}$ depends on y.

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^x + \sin y) = e^x$

$\frac{\partial f/\partial x}{f} = \frac{e^x}{e^x + \sin y}$ depends on y $\Rightarrow$ Not separable

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3. (Solve a Separable Equation)
Given $(x+2)(y+3)y' = \left((x+2)\sec^2(x+1) + x^2 + 2\right)(y+1)^2$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{(y+3)}{(y+1)^2}y' = \sec^2(x+1) + \frac{x^2+2}{x+2}$$

Separated form $\frac{y'}{G(y)} = F(x)$

$$\int \frac{y+3}{(y+1)^2}y'dx = \int\left(\frac{1}{y+1} + \frac{2}{(y+1)^2}\right)y'dx = \ln|y+1| - \frac{2}{y+1} + c_1$$

$$\int \frac{x^2+2}{x+2}dx = \int\left(x-2+\frac{6}{x+2}\right)dx$$

$$= \frac{x^2}{2} - 2x + 6\ln|x+2| + c_2$$

$$\begin{array}{r} x-2 \\ x+2\,\overline{)\,x^2+2} \\ \underline{x^2+2x} \\ -2x+2 \\ \underline{-2x-4} \\ 6 \end{array}$$

$$\boxed{\ln|y+1| - \frac{2}{y+1} = \tan(x+1) + \frac{x^2}{2} - 2x + 6\ln|x+2| + c}$$

maple check ✓

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Scores
4.
5.

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [12:25 lecture]

Version 29.9.2010

Instructions: This in-class exam is 40 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Linear Equations)

(a) [60%] Solve the linear model $8\frac{dx}{dt} = -160 + \frac{16t}{t^2+4}x(t)$, $x(0) = 4$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $5\frac{dy}{dx} + (\tan x)y = 0$.

(c) [20%] Solve $\frac{8}{3}\frac{dy}{dx} + \frac{4}{9}y = -128$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

4(a) $x' - \frac{2t}{t^2+4}x = -20$, $x(0)=4$; $W = e^{\int \frac{-2t\,dt}{t^2+4}} = (t^2+4)^{-1}$

$(Wx)' = -20W \rightarrow Wx = -20\int (t^2+4)^{-1}dt \rightarrow Wx = -20\int \frac{2du}{4u^2+4}$

where $2u=t$. Then $x = [-10\tan^{-1}(t/2)+c](t^2+4)$, $c=1$, and

$\boxed{x(t) = [1-10\tan^{-1}(t/2)](t^2+4)}$

4(b) $y = \frac{c}{e^u}$, $u = \int \frac{1}{5}\tan x\,dx = \frac{1}{5}\ln\left|\frac{1}{\cos x}\right| \Rightarrow \boxed{y = \frac{c}{\sqrt[5]{|\sec x|}}}$

4(c) $y_p = \frac{-128}{4/9} = -(32)(9) = -288$

$y_h = \frac{c}{e^u}$, $u = \int \frac{4/9}{8/3}dx = x/6$ [Prop constant c]

$\boxed{y = -(32)(9) + \frac{c}{e^{x/6}}}$

Use this page to start your solution. Attach extra pages as needed.

maple checks (a)→(c) ✓

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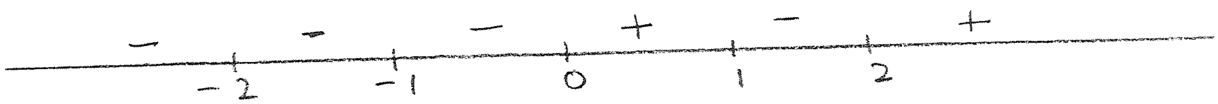
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

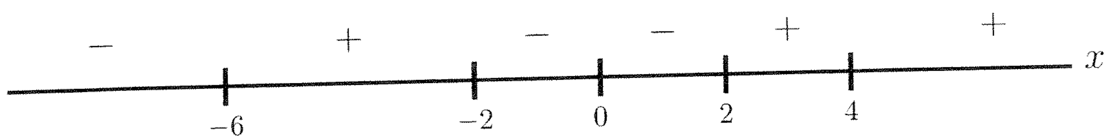
$$\frac{dy}{dx} = e^{2y}\ln(1+(y-1)^2)\,(2-|2-4y|)^3\,(2+y)(4-y^2)(1-y^2)^2.$$

Expected in the phase line diagram are equilibrium points and signs of dy/dx.

equilibria: $y = 1, 0, -2, 2, -1$ Maple check: ✓



(b) [50%] Assume an autonomous equation $y'(x) = f(y(x))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.



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