Differential Equations and Linear Algebra 2250
Midterm Exam 1 [12:25 lecture]
Version 22.9.2010

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)
   (a) [25%] Solve \( y' = \frac{1 + x^3}{1 + x^2} \).
   (b) [25%] Solve \( y' = (\cos x + 1)(\sin x - 1) \).
   (c) [25%] Solve \( y' = e^x \ln(1 + e^x) \), \( y(0) = 1 \).
   (d) [25%] Find the position \( x(t) \) from the velocity model \( \frac{dx}{dt} = 10e^{4t}v(t) = -32e^{4t} \), \( v(0) = -\frac{4}{3} \) and the position model \( \frac{dx}{dt} = v(t) \), \( x(0) = 0 \).

   1(a) \( y' = x + \frac{1}{1 + x^2} - \frac{x}{1 + x^2} \) by the division algorithm
   \[ y = \frac{x^2}{2} + \tan^{-1}(x) - \frac{1}{2} \ln(1 + x^2) + C \]

   1(b) \( y' = \cos x \arctan x - \cos x + \arctan x - 1 \)
   \[ y = -\frac{\arctan^2 x}{2} - \arctan x - \cos x - x + C \]

   1(c) \( y' = \ln u \ du \) \( u = 1 + e^x \), \( du = e^x \ dx \)
   \[ y = u \ln(u) - u + C \]
   \[ y(0) = 1 \Rightarrow 1 = 2 \ln 2 - 2 + C \]
   \[ y = (1 + e^x) \ln(1 + e^x) - (1 + e^x) + 3 - 2 \ln 2 \]

   1(d) \( \int \frac{d}{dt}(10e^{4t}v) dt = -32 \int e^{4t}v \ dt \Rightarrow 10e^{4t}v = -8e^{4t} + C \)
   \[ v(0) = -\frac{4}{3} \Rightarrow 10(1)\left(-\frac{4}{3}\right) = -8(1) + C \Rightarrow C = 0 \]
   \[ v(t) = -\frac{4t}{5} \]
   \[ x(t) = \int v(t) dt = -\frac{4t^2}{10} + C_1 \]
   \[ x(0) = 0 \Rightarrow C_1 = 0 \]
   \[ x(t) = -\frac{4t^2}{10} \]

Use this page to start your solution. Attach extra pages as needed. Maple checks (a) → (d): √
2. (Classification of Equations)

The differential equation $y' = f(x, y)$ is defined to be separable provided $f(x, y) = F(x)G(y)$ for some functions $F$ and $G$.

(a) [40%] Check the problems that can be converted into separable form. No details expected.

<table>
<thead>
<tr>
<th>$y' + xy = y(y + x) + 2xy^2$</th>
<th>$y' = (x - 1)(y + 1) - (1 - x)y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' = e^{x+y} - 3x e^y$</td>
<td>$x^3 y' = x y^2 + x^2 y$</td>
</tr>
</tbody>
</table>

(b) [10%] State a partial derivative test that decides if $y' = f(x, y)$ is a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = x^3 + y$ is linear and not quadrature. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + \sin(y)$ is not separable. Supply all details.

(b) $y' = f(x, y)$ Quadrature $\iff$ $\frac{\partial f}{\partial y} = 0$

(c) Linear $\iff$ $\frac{\partial f}{\partial y}$ is independent of $y$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + y) = 1 \quad \text{independent of } y \quad \Rightarrow \quad \text{Linear}$$

$$\frac{\partial f}{\partial y} = 1 \neq 0 \quad \Rightarrow \quad \text{Not quadrature} \quad \text{used test (b)}$$

(d) $y' = f(x, y)$ is not separable if $\frac{\partial f/\partial x}{f}$ depends on $x$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^x + \sin y) = e^x$$

$$\frac{\partial f/\partial x}{f} = \frac{e^x}{e^x + \sin y} \quad \text{depends on } y \quad \Rightarrow \quad \text{Not separable}$$

Use this page to start your solution. Attach extra pages as needed.
3. (Solve a Separable Equation)

Given \((x + 2)(y + 3)y' = \left((x + 2) \sec^2(x + 1) + x^2 + 2\right)(y + 1)^2\).

Find a non-equilibrium solution in implicit form.
To save time, do not solve for \(y\) explicitly and do not solve for equilibrium solutions.

\[
\frac{(y+3)}{(y+1)^2} \frac{dy}{dx} = \sec^2(x+1) + \frac{x^2+2}{x+2} \quad \text{Separated form} \quad \frac{y'}{G(y)} = f(x)
\]

\[
\int \frac{y+3}{(y+1)^2} \, dy = \int \left(\frac{1}{y+1} + \frac{2}{(y+1)^2}\right) \, dy = \ln|y+1| - \frac{2}{y+1} + c_1
\]

\[
\int \frac{x^2+2}{x+2} \, dx = \int \left(x - 2 + \frac{6}{x+2}\right) \, dx = \frac{x^2}{2} - 2x + 6 \ln|x+2| + c_2
\]

\[
\ln|y+1| - \frac{2}{y+1} = \tan(x+1) + \frac{x^2}{2} - 2x + 6 \ln|x+2| + C
\]

Maple check \(\checkmark\)

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Differential Equations and Linear Algebra 2250
Midterm Exam 1 [12:25 lecture]
Version 29.9.2010

Instructions: This in-class exam is 40 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Linear Equations)

(a) [60%] Solve the linear model \(8 \frac{dx}{dt} = -160 + \frac{16t}{t^2 + 4} x(t)\), \(x(0) = 4\). Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation \(5 \frac{dy}{dx} + (\tan x) y = 0\).

(c) [20%] Solve \(8 \frac{dy}{dx} + 4 \frac{1}{9} y = -128\) using the superposition principle \(y = y_h + y_p\). Expected are answers for \(y_h\) and \(y_p\).

\[4(a)\] \(x' - \frac{2t}{t^2 + 4} x = -20\), \(x(0) = 4\); \(W = \exp\left(-\int \frac{2t}{t^2 + 4} dt\right) = (t^2 + 4)^{-1}\)

\((Wx)' = -20 W \rightarrow Wx = -20 \int (t^2 + 4)^{-1} dt \rightarrow Wx = -20 \int \frac{2u}{u^2 + 4} du\)

where \(u = t\). Then \(x = \left[-10 \tan^{-1}(t/2) + c\right](t^2 + 4)^{-1}, c = 1\), and
\[x(t) = \left[1 - 10 \tan^{-1}(t/2)\right](t^2 + 4)\]

\[4(b)\] \(\gamma = \frac{c}{\exp(u)}\); \(u = \frac{1}{2} \tan^{-1}(x)\) \(\Rightarrow y = \frac{c}{\sqrt{1 + x^2}}\)

\[4(c)\] \(\frac{dy}{dx} = \frac{-128}{9}\); \((32)(9) = -288\)

\(y_h = \frac{c}{\exp(u)}\); \(u = \int \frac{4/9}{8/3} dx = x/6\) [prop. constant \(c\)]

\[y = -(32)(9) + \frac{c}{\exp(x/6)}\]

Use this page to start your solution. Attach extra pages as needed. \(\triangledown\) 

Mark (a) \(\rightarrow\) (c) \(\checkmark\)
5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

\[
\frac{dy}{dx} = e^{2y} \ln(1 + (y - 1)^2) (2 - |2 - 4y|)^3 (2 + y)(4 - y^2)(1 - y^2)^2.
\]

Expected in the phase line diagram are equilibrium points and signs of \(dy/dx\).

\[
equilibrium: \quad y = 1, 0, -2, 2^{-1} \quad \text{maple check: } \checkmark
\]

(b) [50%] Assume an autonomous equation \(y'(x) = f(y(x))\). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

Use this page to start your solution. Attach extra pages as needed.