Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [7:30 lecture]
Version 23.9.2010

Scores
1.
2.
3.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

(a) [25%] Solve $y' = \dfrac{1+x^{-3}}{x^{-1}+x^{-2}}$.

(b) [25%] Solve $y' = (\cos x + 1)(\sin 2x - 1)$.

(c) [25%] Solve $y' = e^{-x}\ln|1+e^{-x}|$, $y(0) = 1$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(25e^{2t}v(t)\right) = -32e^{2t}$, $v(0) = -\frac{16}{25}$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 0$.

1(a) $\dfrac{1+x^{-3}}{x^{-1}+x^{-2}} = \dfrac{x^3+1}{x^2+x} = x - 1 + \dfrac{1}{x}$ by the division algorithm.

$\boxed{y = \dfrac{x^2}{2} - x + \ln|x| + C}$

1(b) $y' = \cos x \sin 2x - \cos x + \sin 2x - 1$
$= 2\cos x \cos x \sin x - \cos x + \sin 2x - 1$

$\boxed{y = -\dfrac{2}{3}\cos^3 x - \sin x - \dfrac{1}{2}\cos 2x - x + C}$

1(c) $y' = -\ln|u|\,du$ where $1+e^{-x} = u$, $-e^{-x}dx = du$

$y = -(u\ln|u| - u) + c$

$y = -(1+e^{-x})\ln|1+e^{-x}| + (1+e^{-x}) + c$

$y(0) = 1 \Rightarrow 1 = -2\ln 2 + 2 + c \Rightarrow c = -1 + 2\ln 2$

$\boxed{y = -(1+e^{-x})\ln|1+e^{-x}| + e^{-x} + 2\ln 2}$

1(d) $\int \frac{d}{dt}(25e^{2t}v)\,dt = -32\int e^{2t}dt \Rightarrow 25e^{2t}v = -16e^{2t} + C$

$v(0) = -\frac{16}{25} \Rightarrow 25(1)\left(\frac{-16}{25}\right) = -16(1) + C \Rightarrow C = 0$

$\boxed{v(t) = \dfrac{-16}{25}}$ Then $x(t) = \int v(t)\,dt = \dfrac{-16}{25}t + C_1$

$x(0) = 0 \Rightarrow C_1 = 0$

$\boxed{x(t) = \dfrac{-16}{25}t}$

maple checks (a) → (d) ✓

Use this page to start your solution. Attach extra pages as needed.

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

☒ $y' + xy = y(2y + x) + x^2y^2$	☒ $y' = (x-1)(y^2+1) - (1-x)y^2$
☒ $y' = e^{x-y} - 3e^x e^{-y}$	☒ $x^3y' = xy + x^2y$

(b) [10%] State a partial derivative test that concludes $y' = f(x, y)$ is not a separable differential equation.

(c) [20%] Apply classification tests to show that $y' + xy = x^3 + y$ is a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + xy$ is not separable. Supply all details.

(b) $\dfrac{\partial f/\partial x}{f}$ depends on y $\Rightarrow$ $y' = f(x,y)$ not separable.

(c) Test: $\dfrac{\partial f}{\partial y}$ independent of y $\Rightarrow$ linear.

$f(x,y) = x^3 + y - xy$ $\Rightarrow$ $\dfrac{\partial f}{\partial y} = 1 - x$ independent of y

$\Rightarrow$ $y' = x^3 + y + (-xy)$ is linear.

(d) Test: see part (b)

$\partial f/\partial x = \dfrac{\partial}{\partial x}(e^x + xy) = e^x + y$

$\dfrac{\partial f/\partial x}{f} = \dfrac{e^x + y}{e^x + xy}$ depends on y $\Rightarrow$ $y' = e^x + xy$ is not separable.

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3. (Solve a Separable Equation)
Given $(x^2+4)(y+3)y' = \left((x^2+4)\csc^2(x) + x^2 + 2\right)(y+4)(y+1)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{y+3}{(y+1)(y+4)}y' = \csc^2 x + \frac{x^2+2}{x^2+4}$$

separated form in $\frac{y'}{G(y)} = F(x)$

$$\frac{y+3}{(y+1)(y+4)} = \frac{2/3}{y+1} + \frac{\frac{-1}{-3}}{y+4}$$

by Heaviside cover-up method

$$\frac{x^2+2}{x^2+4} = 1 + \frac{-2}{x^2+4}$$

by the division algorithm

$$\int \frac{(y+3)y'dx}{(y+1)(y+4)} = \frac{2}{3}\ln|y+1| + \frac{1}{3}\ln|y+4| + c_1$$

$$\int \frac{x^2+2}{x^2+4}dx = x - \tan^{-1}(x/2) + c_2$$

$$\boxed{\frac{2}{3}\ln|y+1| + \frac{1}{3}\ln|y+4| = x - \tan^{-1}(x/2) - \tan x + c}$$

maple check: ✓

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Name KEY

Scores
4.
5.

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [7:30 lecture]
Version 30.9.2010

Instructions: This in-class exam is 40 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. **(Linear Equations)**

(a) [60%] Solve the linear model $4\frac{dx}{dt} = -32 + \frac{4}{t^2+t}x(t)$, $x(1) = -4$. Show all integrating factor steps.

Hint: Partial fraction theory implies $\frac{1}{t^2+t} = \frac{1}{t} - \frac{1}{t+1}$.

(b) [20%] Solve the homogeneous equation $3\frac{dy}{dx} - (\cot x)y = 0$.

(c) [20%] Solve $\frac{11}{3}\frac{dy}{dx} + \frac{1}{3}y = -121$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

4(a) $x' - \frac{1}{t(t+1)}x = -8$, $x(1) = -4$; $W = e^u$, $u = \int \frac{-dt}{t(t+1)}$

$u = \int \frac{dt}{t+1} - \int \frac{dt}{t} = \ln\left|\frac{t+1}{t}\right| + c \Rightarrow \boxed{W = \frac{t+1}{t}}$

$(xW)' = -8W \rightarrow xW = -8\int \frac{t+1}{t}dt \rightarrow xW = -8(t + \ln|t|) + c$

$\rightarrow x = \frac{t}{t+1}(-8t - 8\ln|t| + c) \rightarrow -4 = \frac{1}{2}(-8 + 0 + c) \rightarrow c = 0$

$\boxed{x(t) = \frac{t}{t+1}(-8t - 8\ln|t|)}$ maple check: ✓

4(b) $y = \frac{c}{e^u}$, $u = \int -\frac{1}{3}\cot x\,dx = -\frac{1}{3}\ln|\sin x| + c$

$\boxed{y = \frac{c}{(\csc x)^{1/3}}}$ maple check: ✓

4(c) $y_p = \frac{-121}{1/3} = -363$, $y_h = \frac{c}{e^u}$, $u = \int \frac{1/3}{11/3}dx = x/11$

$\boxed{y = -363 + \frac{c}{e^{x/11}}}$ maple check: ✓

Use this page to start your solution. Attach extra pages as needed.

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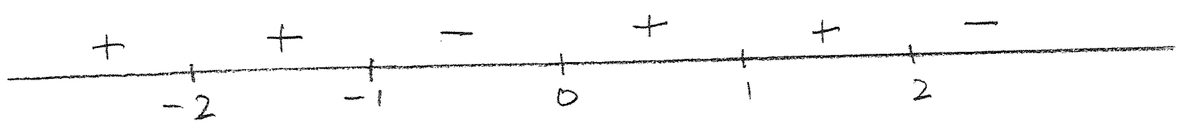
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

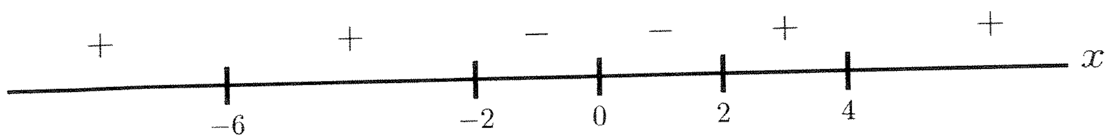
$$\frac{dy}{dx} = e^{-2y}\ln(1+(y-2)^2)\,(2-|2-4y|)^3\,(2+y)(4-y^2)(1-y^2)^3.$$

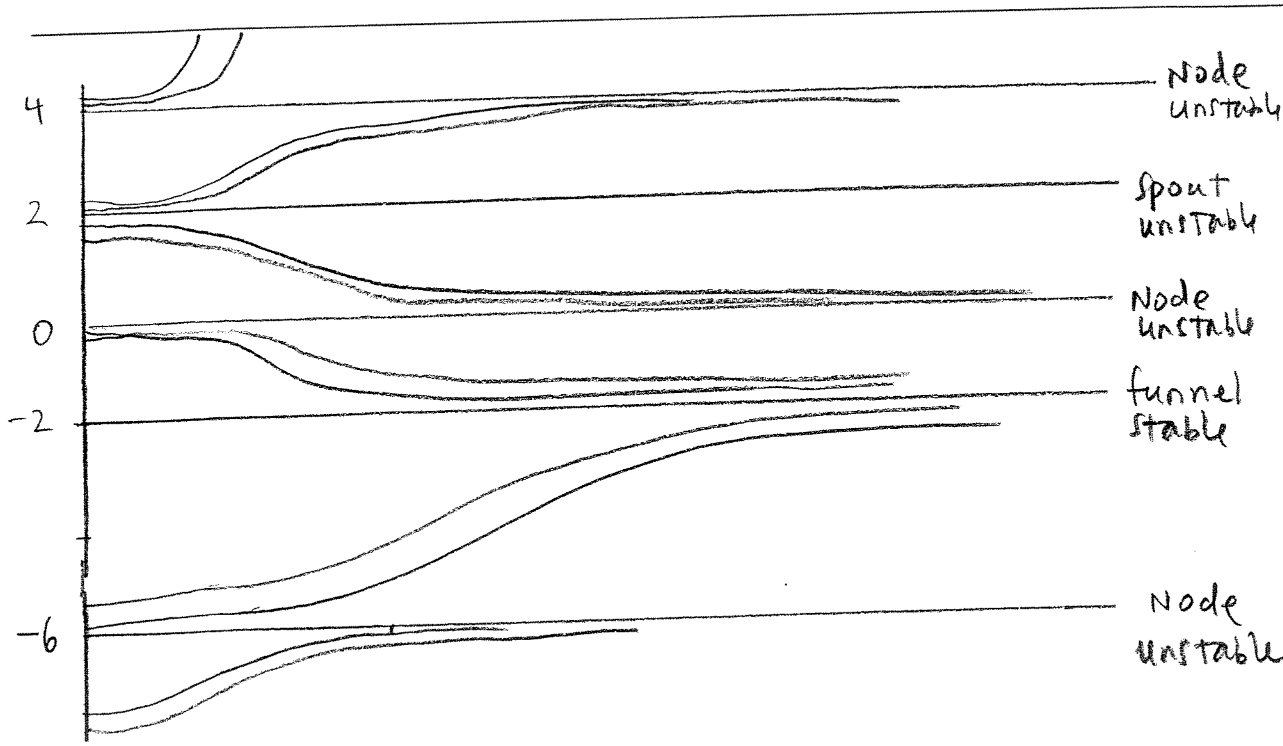
Expected in the phase line diagram are equilibrium points and signs of dy/dx.

equilibria: $y = 2, 1, 0, -2, -1$ maple check: ✓



(b) [50%] Assume an autonomous equation $y'(x) = f(y(x))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.





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