Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)
   (a) [25%] Solve \( y' = \frac{1 + x^{-3}}{x^{-1} + x^{-2}} \).
   (b) [25%] Solve \( y' = (\cos x + 1)(\sin 2x - 1) \).
   (c) [25%] Solve \( y' = e^{-x} \ln |1 + e^{-x}|, y(0) = 1 \).
   (d) [25%] Find the position \( x(t) \) from the velocity model \( \frac{dx}{dt} = 25e^{2t}v(t) = -32e^{2t}, v(0) = -\frac{16}{25} \) and the position model \( \frac{dx}{dt} = v(t), x(0) = 0 \).

1(a) \( \frac{1 + x^{-3}}{x^{-1} + x^{-2}} = \frac{x^{3} + 1}{x^{2} + x} = x - 1 + \frac{1}{x} \) by \( \text{division algorithm} \).
   \( y = \frac{x^2}{2} - x + \ln|\sqrt{x} + C| \)

1(b) \( y' = \cos x \sin 2x - \cos x + \sin 2x - 1 \)
   \( = 2 \cos x \cos x \sin x - \cos x + \sin 2x - 1 \)
   \( y = -\frac{2}{2} \cos^2 x - \sin x - \frac{1}{2} \cos 2x - x + C \)

1(c) \( y' = -\frac{e^x}{u} du \) where \( 1 + e^x = u \), \( -e^x dx = du \)
   \( y = -(u \ln|u| - u) + C \)
   \( y = -(1 + e^x) \ln|1 + e^x| + (1 + e^x) + C \)
   \( y(0) = 1 \Rightarrow 1 = -2 \ln 2 + 2 + C \Rightarrow C = -1 + 2 \ln 2 \)
   \( y = -(1 + e^x) \ln|1 + e^x| + e^x + 2 \ln 2 \)

1(d) \( \int \frac{d}{dt} (25e^{2t}v) dt = -32 \int e^{2t} dt \Rightarrow 25e^{2t}v = -16e^{2t} + C \)
   \( v(0) = -\frac{16}{25} \Rightarrow 25(1)(-\frac{16}{25}) = -16(1) + C \Rightarrow C = 0 \)
   \( v(t) = -\frac{16}{25} \)
   Then \( \chi(t) = \int v(t) dt = -\frac{16}{25}t + C_1 \)
   \( \chi(0) = 0 \Rightarrow C_1 = 0 \)

\( \chi(t) = -\frac{16}{25}t \)

Use this page to start your solution. Attach extra pages as needed.
2. (Classification of Equations)

The differential equation \( y' = f(x, y) \) is defined to be _separable_ provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

(a) [40%] Check the problems that can be converted into separable form. No details expected.

\[
\begin{array}{|c|c|}
\hline
y' + xy = y(2y + x) + x^2 y^2 & y' = (x - 1)(y^2 + 1) - (1 - x)y^2 \\
\hline
y' = e^{x-y} - 3e^x e^{-y} & x^3 y' = xy + x^2 y \\
\hline
\end{array}
\]

(b) [10%] State a partial derivative test that concludes \( y' = f(x, y) \) is not a separable differential equation.

(c) [20%] Apply classification tests to show that \( y' + xy = x^3 + y \) is a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that \( y' = e^x + xy \) is not separable. Supply all details.

(b) \( \frac{\partial f}{\partial x} \) depends on \( y \) \( \Rightarrow \) \( y' = f(x, y) \) _not separable_.

(c) Test: \( \frac{\partial f}{\partial y} \) independent of \( y \) \( \Rightarrow \) linear.

\[
f(x, y) = x^3 + y - xy \Rightarrow \frac{\partial f}{\partial y} = 1 - x \text{ independent of } y. \\
y' = x^3 + y + (-xy) \text{ is linear.}
\]

(d) Test: See part (b)

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^x + xy) = e^x + y \\
\frac{\partial f}{\partial x} = \frac{e^x + y}{e^{x+y}} \text{ depends on } y \Rightarrow y' = e^x + xy \text{ is not separable.}
\]

Use this page to start your solution. Attach extra pages as needed.
3. (Solve a Separable Equation)
Given \((x^2 + 4)(y + 3)y' = (x^2 + 4) \csc^2(x) + x^2 + 2 \) \((y + 4)(y + 1)\).

Find a non-equilibrium solution in implicit form.
To save time, do not solve for \(y\) explicitly and do not solve for equilibrium solutions.

\[
\frac{y+3}{(y+1)(y+4)} \frac{y'}{y+4} = \csc^2 x + \frac{x^2+2}{x^2+y} \quad \text{separated for } m
\]

\[
\frac{y+3}{(y+1)(y+4)} = \frac{2/3}{y+1} - \frac{1/3}{y+4} \quad \text{by Heaviside level-up method}
\]

\[
\frac{x^2+2}{x^2+y} = 1 - \frac{2}{x^2+y} \quad \text{by Re division algorithm}
\]

\[
\int \frac{(y+3)y'dx}{(y+1)(y+4)} = \frac{2}{3} \ln |y+1| + \frac{1}{3} \ln |y+4| + c_1
\]

\[
\int \frac{x^2+2}{x^2+y} dx = x - \tan^{-1}(x/2) + c_2
\]

\[
\frac{2}{3} \ln |y+1| + \frac{1}{3} \ln |y+4| = x - \tan^{-1}(x/2) - \tan x + c
\]

Maple check: √
Differential Equations and Linear Algebra 2250
Midterm Exam 1 [7:30 lecture]
Version 30.9.2010

Instructions: This in-class exam is 40 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

4. (Linear Equations)
(a) [60%] Solve the linear model \( \frac{dx}{dt} = -32 + \frac{4}{t^2 + l} x(t) \), \( x(1) = -4 \). Show all integrating factor steps.
Hint: Partial fraction theory implies \( \frac{1}{t^2 + l} = \frac{1}{t} - \frac{1}{t + 1} \).
(b) [20%] Solve the homogeneous equation \( 3 \frac{dy}{dx} + (\cot x)y = 0 \).
(c) [20%] Solve \( \frac{11}{3} \frac{dy}{dx} + \frac{1}{3} y = -121 \) using the superposition principle \( y = y_h + y_p \). Expected are answers for \( y_h \) and \( y_p \).

4(a) \( x' - \frac{1}{t(t+1)} x = -8 \), \( x(1) = -4 \), \( W = e^u \), \( u = \int \frac{dt}{t(t+1)} \)
\( u = \ln \left| \frac{t+1}{t} \right| + c \Rightarrow W = \frac{t+1}{t} \)
\( (xW)' = -8W \rightarrow xW = -8 \int \ln(t+1) dt \rightarrow xW = -8(t \ln(t+1) + C) \rightarrow -4 = \frac{1}{2} (-8 + 0 + C) \rightarrow C = 0 \)
\( x(t) = \frac{t}{t+1} \left( -8 t - 8 \ln(t+1) \right) \)

4(b) \( y = \frac{c}{e^u} \), \( u = \int \frac{-1}{2} \cot x \, dx = -\frac{1}{2} \ln |\sin x| + c \)
\( y = \frac{c}{(\csc x)^{\frac{1}{2}}} \)

4(c) \( y_p = \frac{-121}{1/2} = -362 \), \( y_h = \frac{c}{e^u} \), \( u = \int \frac{1/3}{11/3} dx = x/11 \)
\( y = -362 + \frac{c}{e^{x/11}} \)

Use this page to start your solution. Attach extra pages as needed.
5. (Stability)
   (a) [50%] Draw a phase line diagram for the differential equation
   \[
   \frac{dy}{dx} = e^{-2y} \ln(1 + (y - 2)^2) (2 - |2 - 4y|)^3 (2 + y)(4 - y^2)(1 - y^2)^3.
   \]
   Expected in the phase line diagram are equilibrium points and signs of \(dy/dx\).
   \[\text{equilibria: } \quad y = 2, 1, 0, -2, -1 \quad \text{maple check: } \checkmark\]

   (b) [50%] Assume an autonomous equation \(y'(x) = f(y(x))\). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

Use this page to start your solution. Attach extra pages as needed.