KEY

Name ________________ ________________

Scores
1. 100
2. 100
3. 100p

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [12:25 lecture]

Version 20.9.2010 Monday week 5

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

100

(a) [25%] Solve $y' = \dfrac{1+x^2}{1+x}$.

(b) [25%] Solve $y' = (\tan x + 1)(\tan x - 1)$.

(c) [25%] Solve $y' = x^3 \ln\left|1+x^4\right|$, $y(0) = 1$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(e^{2t}v(t)\right) = -\dfrac{16}{5}e^{2t}$, $v(0) = \frac{2}{5}$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = -101$.

a) A

$y' = \dfrac{1+x^2}{1+x}$

Division algorithm: $x+1 \,\overline{)\, x^2+1}$ quotient $x-1$ r. 2

$$\begin{array}{r} x^2+x \\ \hline 1-x \\ -1-x \\ \hline 2 \end{array}$$

$y' = x - 1 + \dfrac{2}{1+x}$

$\int y'\,dx = \int x - 1 + \dfrac{2}{1+x}\,dx$

$\boxed{y = \dfrac{x^2}{2} - x + 2\ln|1+x| + C}$

b) A

$y' = (\tan x+1)(\tan x-1)$

$y' = \tan^2 x - 1$

$\int y'\,dx = \int \tan^2 x - 1\,dx$

$y = -x + \tan x - x + C$

$\boxed{y = \tan x - 2x + C}$

c) $y' = x^3 \ln|1+x^4|$, $y(0)=1$

$u = 1+x^4$

$du = 4x^3 dx$

$\int y'\,dx = \frac{1}{4}\int \ln|u|\,du$

$y = \frac{1}{4}(u\ln u - u + C)$

$y = \frac{1}{4}(1+x^4)\ln|1+x^4| - \frac{1}{4}(1+x^4) + C$

use initial condition:

$1 = \frac{1}{4}(1)\underbrace{\ln|1|}_{0} - \frac{1}{4} + C$

$1 = -\frac{1}{4} + C$

$C = 1.25 = 5/4$

A $\boxed{y = \frac{1}{4}(1+x^4)\ln|1+x^4| - \frac{1}{4}(1+x^4) + \frac{5}{4}}$

Use this page to start your solution. Attach extra pages as needed.

1, d)

$$\frac{d}{dt}\left(e^{2t}v(t)\right) = \frac{-16}{5}e^{2t} \qquad v(0) = \frac{2}{5} \qquad x(0) = -101$$

$$\int \frac{d}{dt}\left(e^{2t}v(t)\right)dt = \frac{-16}{5}\int e^{2t}\,dt$$ Quadrature.

$$e^{2t}v(t) = \frac{-16}{5 \cdot 2}e^{2t} + C$$

$$v(t) = \frac{-8e^{2t}}{5e^{2t}} + \frac{C}{e^{2t}}$$

$$v(t) = \frac{-8}{5} + Ce^{-2t}$$

Solve for C using initial condition $v(0) = \frac{2}{5}$

$$\frac{2}{5} = -\frac{8}{5} + Ce^{0}$$

$$\frac{10}{5} = C$$

$$2 = C$$

$$v(t) = -\frac{8}{5} + 2e^{-2t}$$ Exact solution for $v(t)$.

$$v(t) = x'(t)$$

$$\int x'(t)dt = \int -\frac{8}{5} + 2e^{-2t}\,dt$$ Quadrature.

$$x(t) = \frac{-8}{5}t + \frac{2}{-2}e^{-2t} + C$$

Solve for C using initial condition $x(0) = -101$.

$$-101 = \frac{-8}{5} \cdot 0 - e^{0} + C$$

$$-101 = -1 + C$$

A $$-100 = C$$

$$\boxed{x(t) = \frac{-8}{5}t + -e^{-2t} - 100}$$ Exact solution for $x(t)$

Name.

A
100

$\frac{\partial F}{\partial y} = 0 \Rightarrow$ Quad

$\frac{\partial F}{\partial y}$ ind. of $y \Rightarrow$ Linear

$\frac{x(x,y)}{F}$ ind. of $y \Rightarrow$ Separable

2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

$\boxed{X}$	$y' + xy = y(y + x) - 2y^2$ +10%	$\boxed{X}$	$y' = (x-1)(y+1) + (1-x)y$ +10%
$\boxed{X}$	$y' = \cos(x + y) - \cos(x)\cos(y)$ +10%	$\boxed{X}$	$x^3y' = xy + x^2y$ +10%

(b) [10%] State a partial derivative test that decides if $y' = f(x, y)$ is a linear differential equation.

(c) [20%] Apply classification tests to show that $y' = x^3 + y^3$ is not quadrature and not linear. Supply all details.

(d) [30%] Apply a test to show that $y' = e^x + \ln|y|$ is not separable. Supply all details.

a)

- $y' = y^2 + xy - 2y^2 - xy$
  $y' = -y^2$
  $\frac{F_x}{F} = \frac{-y^2}{-y^2} = 1 \rightarrow$ sep.

- $y' = (x-1)(y+1) + (1-x)y$
  $= \cancel{xy} - \cancel{y} + x - 1 + \cancel{y} - \cancel{xy}$
  $y' = x - 1$
  $\frac{F_x}{F} = \frac{1}{x-1}$ indep. of $y \Rightarrow$ Sep.

- $y' = \cos(x+y) - \cos(x)\cos(y)$
  $y' = \cancel{\cos x \cos y} - \sin x \sin y - \cancel{\cos x \cos y}$
  $y' = -\sin x \sin y \rightarrow$ sep.

- $x^3 y' = xy + x^2 y$
  $y' = \frac{xy + x^2y}{x^3} = \frac{y}{x^2} + \frac{y}{x} = y\left(\frac{1}{x^2} + \frac{1}{x}\right)$
  $\frac{F_x}{F} = \frac{-2\cancel{y}x^{-3} - \cancel{y}x^{-2}}{\cancel{y}x^{-2} + \cancel{y}x^{-1}} \Rightarrow$ sep.

b) $\frac{\partial F}{\partial y}$ must be independent of y if $y' = F(x,y)$ is linear. +10%

Use this page to start your solution. Attach extra pages as needed.

$\boxed{2c}$ $y' = x^3 + y^3$

Quadrature test:

$\frac{\partial F}{\partial y} = 3y^2$ $\quad$ $\frac{\partial F}{\partial y}$ must $= 0$ to be quadrature, so $y' = x^3 + y^3$ is <u>not</u> quadrature.

+10%

Linear test:

$\frac{\partial F}{\partial y} = 3y^2$ $\quad$ $\frac{\partial F}{\partial y}$ must be independent of y to be linear, so $y' = x^3 + y^3$ is <u>not</u> linear.

+10%

$\boxed{2d}$ $y' = e^x + \ln|y|$

Separable test:

$\frac{F_x(x,y)}{F}$ must be independent of y to be separable

$\frac{F_x(x,y)}{F} = \frac{e^x}{e^x + \ln|y|}$ → is dependent on y, so $y' = e^x + \ln|y|$ is <u>not</u> separable.

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3. (Solve a Separable Equation)

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A

Given $(x+1)(y+2)y' = \left((x+1)\sin(x+1) + x^2 + 1\right)(y+1)(y-1)$.

Find a non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{(y+2)}{(y+1)(y-1)} y' = \sin(x+1) + \frac{x^2+1}{x+1}$$

$$\int \frac{(y+2)}{(y+1)(y-1)} y' dx = \int \sin(x+1) + \frac{x^2+1}{x+1} dx$$

+30%

$$LHS = \int \frac{(y+2)}{(y+1)(y-1)} y' dx$$

$$= \int \left(\frac{A}{y+1} + \frac{B}{y-1}\right) dy \longrightarrow A(y-1) + B(y+1) = y+2$$

$$y=1 \to B = \frac{3}{2}$$

$$y=-1 \to A = \frac{1}{-2}$$

$$= \frac{1}{2}\int \frac{-1}{y+1} + \frac{3}{y-1} dy$$

+30% $$= \frac{1}{2}\left(-\ln|y+1| + 3\ln|y-1|\right) + c$$

$$RHS = \int \sin(x+1) + \frac{x^2+1}{x+1} \longrightarrow$$

$$\begin{array}{r} x-1 \text{ r. } 2 \\ x+1 \,\overline{)\, x^2+1} \\ x^2+x \\ \hline 1-x \\ -1-x \\ \hline 2 \end{array}$$

$$= -\cos(x+1) + \int x - 1 + \frac{2}{x+1} dx$$

+30% $$= -\cos(x+1) + \frac{x^2}{2} - x + 2\ln|x+1| + C$$

Answer:

$$\boxed{\frac{1}{2}\left(-\ln|y+1| + 3\ln|y-1|\right) = -\cos(x+1) + \frac{x^2}{2} - x + 2\ln|x+1| + C}$$

+10%

Well done! Clear notation.

Use this page to start your solution. Attach extra pages as needed.

Name __

Scores
4. 100
5. 100

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [12:25 lecture]
Version 27.9.2010 Monday week 6

Instructions: This in-class exam is 40 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

A
100

4. **(Linear Equations)**

(a) [60%] Solve the linear model $4\frac{dx}{dt} = -160 + \frac{8t}{t^2+1}x(t)$, $x(0) = 0$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $3\frac{dy}{dx} + (\sec^2 x)y = 0$.

(c) [20%] Solve $\frac{5}{3}\frac{dy}{dx} + \frac{16}{9}y = -32$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

a) $4\frac{dx}{dt} - \frac{8t}{t^2+1}x(t) = -160$

$\frac{dx}{dt} - \frac{2t}{t^2+1}x(t) = \frac{-160}{4}$ $\quad \left(\frac{160}{4}\right) = 40$ Put in integrating factor form.

$W = e^{\int \frac{-2t}{t^2+1}dt}$ Find integrating factor, W

$u = t^2+1$

$du = 2t\,dt$

$W = e^{-\int \frac{1}{u}du}$

$W = e^{-\ln u}$

$= u^{-1}$

$W = (t^2+1)^{-1}$

$\frac{(xW)'}{W} = -40$

$\int (xW)'dx = -40\int \frac{1}{t^2+1}dt$

$x(t^2+1)^{-1} = -40\tan^{-1}(t) + C$

$x(t) = \frac{-40\tan^{-1}(t)}{(t^2+1)^{-1}} + \frac{C}{(t^2+1)^{-1}}$

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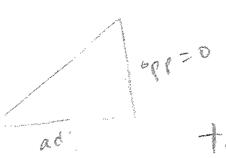
$0 = \frac{-40\tan^{-1}(0)}{(1)^{-1}} + \frac{C}{(1)^{-1}}$ use $x(0) = 0$



$\tan^{-1}(0) = 0$, so $C = 0$, $\rightarrow \boxed{x(t) = -40\tan^{-1}(t)(t^2+1)}$

b) $3\frac{dy}{dx} + (\sec^2 x)y = 0$

$$y = \frac{c}{\text{intg. factor}}$$

$$\frac{dy}{dx} + \frac{(\sec^2 x)}{3} y = 0$$

$$W = e^{\frac{1}{3}\int \sec^2 x\, dx} = e^{\frac{1}{3}\tan x}$$

$$\boxed{y = \frac{c}{e^{1/3 \tan x}}}$$

c) $\frac{5}{3}\frac{dy}{dx} + \frac{16}{9}y = -32 \longrightarrow \frac{dy}{dx} + \frac{16}{9}\cdot\frac{3}{5}y = -32\cdot\frac{3}{5}$

$$y_p = \frac{-32\cdot 9}{16} = -2\cdot 9 = -18$$

$$y_p = -18$$

$$\frac{dy}{dx} + \frac{48}{45}y = \frac{-96}{5}$$

$$y_h = \frac{c}{e^{\int \frac{48}{45}dx}} = \frac{c}{e^{48/45 x}}$$

$$\boxed{y = \frac{c}{e^{48/45 x}} - 18}$$

Name. ____ ________

100p A   Well done!

5. **(Stability)**

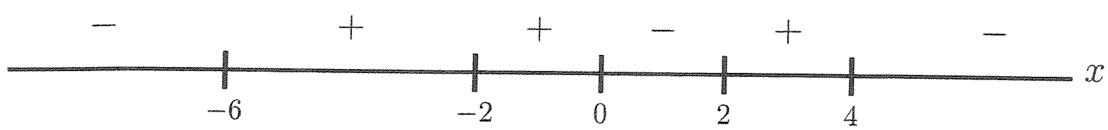
A **(a)** [50%] Draw a phase line diagram for the differential equation

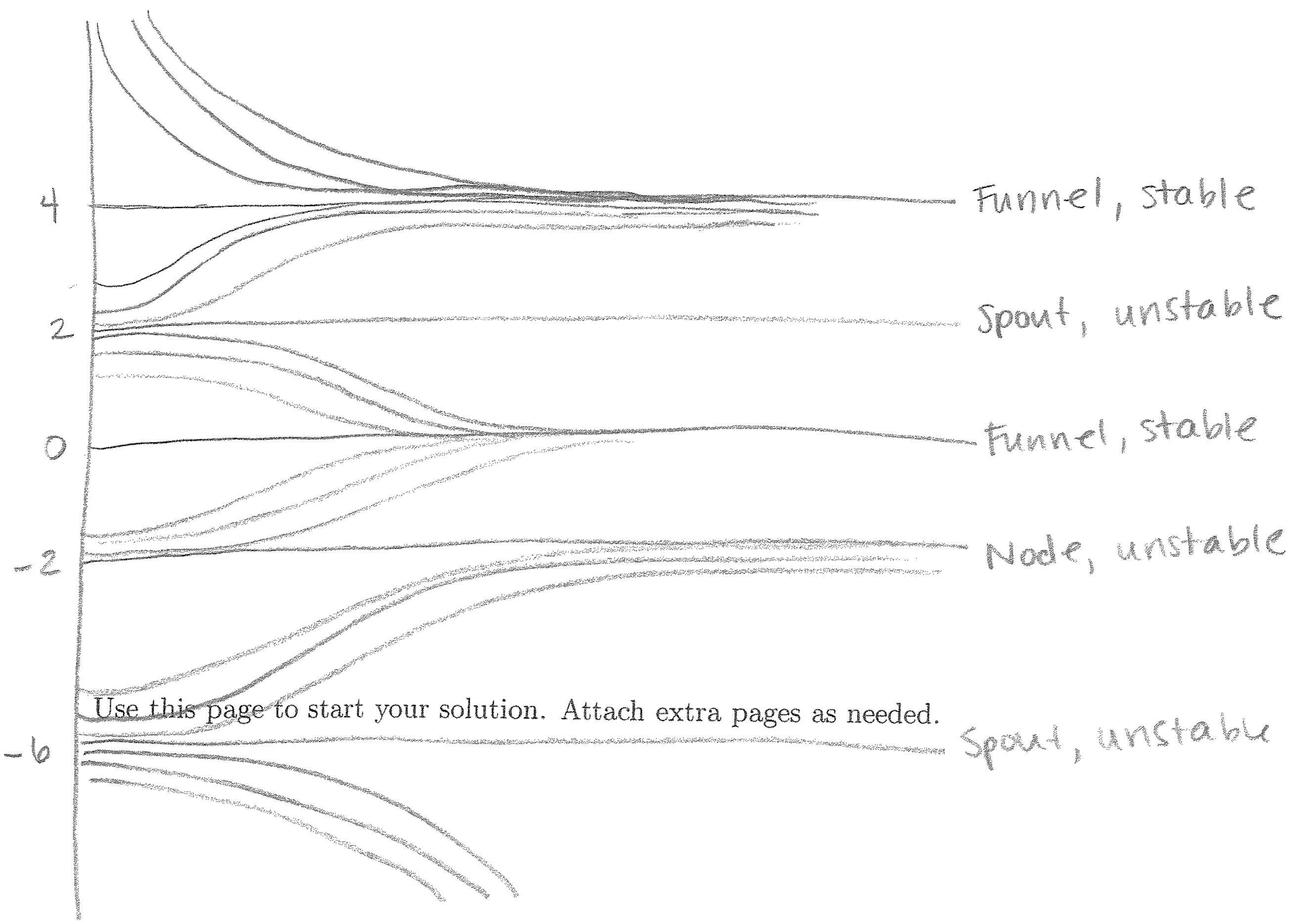
$$\frac{dy}{dx} = e^{2y}\ln(1+y^2)\,(1-|3y-2|)^3\,(2-y)(4-y^2)(1-y^2)^2.$$

Expected in the phase line diagram are equilibrium points and signs of dy/dx.

See attached page.

A **(b)** [50%] Assume an autonomous equation $y'(x) = f(y(x))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: **funnel**, **spout**, **node** [neither spout nor funnel], **stable**, **unstable**.





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5a $e^{2y} \quad \ln(1+y^2)\ (1-|3y-2|)^3 (2-y)(4-y^2)(1-y^2)^2$

$e^{2y} \rightarrow$ no roots

$\ln(1+y^2) \rightarrow 1 = 1+y^2$, $y=0$

$(1-|3y-2|)^3 \rightarrow 1-|3y-2| = 0$

$|3y-2| = 1$

$3y-2=1$

$3y=3$

$y=1$

or

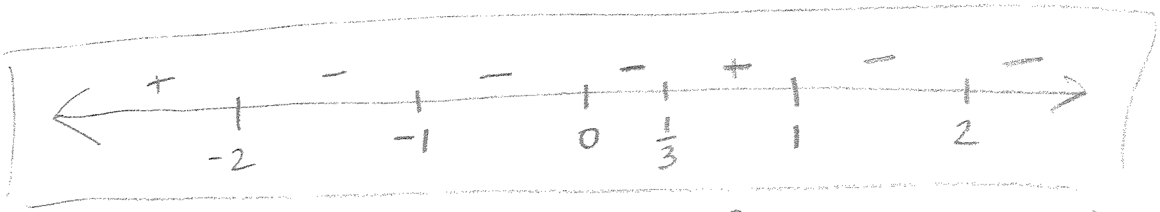
$3y-2=-1$

$3y=1$

$y=\frac{1}{3}$

$y=2 \quad y=\pm 2 \quad y=\pm 1$

roots $= 0, \frac{1}{3}, 1, 2, -2, -1$



$y=-3 \rightarrow e^{-6}$ (+) $\quad \ln(>1)$ (+) $\quad (1-|-9-2|)^3$ (−) $\quad (2+3)$ (+) $\quad (4-9)$ (−) $\quad (1-9)^2$ (+) $=$ (+)

$y=-1.5 \rightarrow$ (+) $\quad \ln(>1)$ (+) $\quad (1-|-4.5-2|)^3$ (−) $\quad (2+1.5)$ (+) $\quad (4-1.5^2)$ (+) $\quad (1-1.5^2)^2$ (+) $=$ (−)

$y=-0.5 \rightarrow$ (+) $\quad$ (+) $\quad (1-|-1.5-2|)^3$ (−) $\quad (2+0.5)$ (+) $\quad (4-0.5^2)$ (+) $\quad (1-0.5^2)^2$ (+) $=$ (−)

$y=\frac{1}{6} \rightarrow$ (+) $\quad$ (+) $\quad (1-|\frac{1}{2}-2|)^3$ (−) $\quad (2-\frac{1}{6})$ (+) $\quad (4-\frac{1}{6}^2)$ (+) $\quad (1-\frac{1}{6}^2)^2$ (+) $=$ (−)

$y=0.5 \rightarrow$ (+) $\quad$ (+) $\quad (1-|1.5-2|)^3$ (+) $\quad (2-0.5)$ (+) $\quad (4-0.5^2)$ (+) $\quad (1-0.5^2)^2$ (+) $=$ (+)

$y=1.5 \rightarrow$ (+) $\quad$ (+) $\quad (1-|4.5-2|)^3$ (−) $\quad (2-1.5)$ (+) $\quad (4-1.5^2)$ (+) $\quad (1-1.5^2)^2$ (+) $=$ (−)

$y=3 \rightarrow$ (+) $\quad$ (+) $\quad (1-|9-2|)^3$ (−) $\quad (2-3)$ (−) $\quad (4-3^2)$ (−) $\quad (1-9)^2$ (+) $=$ (−)