Due date: Submit these problems by the day after classes end. Records are locked on that date and only corrected, never appended. Credits earned here apply only to chapter 7 and not to any other chapter.

Submitted work. Please submit one stapled package per problem. Kindly label problems [Extra Credit]. Label each problem with its corresponding problem number, e.g., [Xc7.1-8]. You may attach this printed sheet to simplify your work.

Problem Xc7.1-8. (Transform to a first order system)
Use the position-velocity substitution \( u_1 = x(t), u_2 = x'(t), u_3 = y(t), u_4 = y'(t) \) to transform the system below into vector-matrix form \( \mathbf{u}'(t) = \mathbf{A} \mathbf{u}(t) \). Do not attempt to solve the system.

\[
\begin{align*}
  x'' - 2x' + 5y &= 0, \\
y'' + 2y' - 5x &= 0.
\end{align*}
\]

Problem Xc7.1-20a. (Dynamical systems)
Prove this result for system
\[
\begin{align*}
x' &= ax + by, \\
y' &= cx + dy.
\end{align*}
\]

Theorem. Let \( \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and define \( \text{trace}(\mathbf{A}) = a + d \). Then \( p_1 = -\text{trace}(\mathbf{A}), p_2 = \det(\mathbf{A}) \) are the coefficients in the determinant expansion
\[
\det(\mathbf{A} - r\mathbf{I}) = r^2 + p_1 r + p_2
\]
and \( x(t) \) and \( y(t) \) in equation (1) are both solutions of the differential equation \( u'' + p_1 u' + p_2 u = 0 \).

Problem Xc7.1-20b. (Solve dynamical systems)
(a) Apply the previous problem to solve
\[
\begin{align*}
x' &= 2x - y, \\
y' &= x + 2y.
\end{align*}
\]
(b) Use first order methods to solve the system
\[
\begin{align*}
x' &= 2x - y, \\
y' &= 2y.
\end{align*}
\]

Problem Xc7.2-12. (General solution answer check)
(a) Verify that \( \mathbf{x}_1(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( \mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \) are solutions of \( \mathbf{x}' = \mathbf{A} \mathbf{x} \), where
\[
\mathbf{A} = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}.
\]
(b) Apply the Wronskian test \( \det(\text{aug}(\mathbf{x}_1, \mathbf{x}_2)) \neq 0 \) to verify that the two solutions are independent.
(c) Display the general solution of \( \mathbf{x}' = \mathbf{A} \mathbf{x} \).

Extra credit problems chapter 7 continue on the next page.
Problem Xc7.2-14. (Particular solution)
(a) Find the constants $c_1$, $c_2$ in the general solution
\[ x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} \]
satisfying the initial conditions $x_1(0) = 4$, $x_2(0) = -1$.
(b) Find the matrix $A$ in the equation $x' = Ax$. Use the formula $AP = PD$ and Fourier’s model for $A$, which is given implicitly in (a) above.

Problem Xc7.3-8. (Eigenanalysis method $2 \times 2$)
(a) Find $\lambda_1$, $\lambda_2$, $v_1$, $v_2$ in Fourier’s model $A (c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$ for
\[ A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} . \]
(b) Display the general solution of $x' = Ax$.

Problem Xc7.3-20. (Eigenanalysis method $3 \times 3$)
(a) Find $\lambda_1$, $\lambda_2$, $\lambda_3$, $v_1$, $v_2$, $v_3$ in Fourier’s model $A (c_1 v_1 + c_2 v_2 + c_3 v_3) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3$ for
\[ A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix} . \]
(b) Display the general solution of $x' = Ax$.

Problem Xc7.3-30. (Brine Tanks)
Consider two brine tanks satisfying the equations
\[ x_1'(t) = -k_1 x_1 + k_2 x_2, \quad x_2' = k_1 x_1 - k_2 x_2. \]
Assume $r = 10$ gallons per minute, $k_1 = r/V_1$, $k_2 = r/V_2$, $x_1(0) = 30$ and $x_2(0) = 0$. Let the tanks have volumes $V_1 = 50$ and $V_2 = 25$ gallons. Solve for $x_1(t)$ and $x_2(t)$.

Problem Xc7.3-40. (Eigenanalysis method $4 \times 4$)
Display (a) Fourier’s model and (b) the general solution of $x' = Ax$ for the $4 \times 4$ matrix
\[ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 21 & 5 & 27 & 9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix} . \]

End of extra credit problems chapter 7.