Matrix Exponential: Putzer Formula
Variation of Parameters for Systems
Undetermined Coefficients for Systems

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The $2 \times 2$ Matrix Exponential $e^{At}$

The matrix $e^{At}$ has columns equal to the solutions of the two problems

$$\begin{cases}
\vec{u}_1'(t) = A\vec{u}_1(t), \\
\vec{u}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{cases}$$

$$\begin{cases}
\vec{u}_2'(t) = A\vec{u}_2(t), \\
\vec{u}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{cases}$$

Briefly, the matrix $\Phi(t) = e^{At}$ satisfies the two conditions

1. $\Phi'(t) = A\Phi(t)$,
2. $\Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. 
Putzer Formula for $2 \times 2$ Matrices

\[ e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} (A - \lambda_1 I) \quad \text{if } A \text{ is } 2 \times 2, \lambda_1 \neq \lambda_2 \text{ real.} \]

\[ e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t} (A - \lambda_1 I) \quad \text{if } A \text{ is } 2 \times 2, \lambda_1 = \lambda_2 \text{ real.} \]

\[ e^{At} = e^{at} \cos bt I + \frac{e^{at} \sin bt}{b} (A - aI) \quad \text{if } A \text{ is } 2 \times 2, \lambda_1 = \bar{\lambda}_2 = a + ib, \quad b > 0. \]
How to Remember Putzer’s $2 \times 2$ Formula

The expressions

\[ e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I), \]
\[ r_1(t) = e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \]

are enough to generate all three formulas. Fraction $r_2$ is the $d/d\lambda$-Newton quotient for $r_1$. It has limit $te^{\lambda_1 t}$ as $\lambda_2 \to \lambda_1$, therefore the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \lambda_2 = a + ib$ with $b > 0$, then the fraction $r_2$ is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

\[ r_2(t) = \frac{z - \bar{z}}{w - \bar{w}} = \frac{\sin bt}{b}. \]

Taking real parts of expression (1) gives the complex case formula.
Theorem 1 (Variation of Parameters for Systems)
Let $A$ be a constant $n \times n$ matrix and $F(t)$ a continuous function near $t = t_0$. The unique solution $x(t)$ of the matrix initial value problem

$$x'(t) = Ax(t) + F(t), \quad x(t_0) = x_0,$$

is given by the variation of parameters formula

$$x(t) = e^{At}x_0 + e^{At} \int_{t_0}^{t} e^{-rA}F(r)dr.$$
Undetermined Coefficients

Theorem 2 (Polynomial solutions)

Let $f(t)$ be a polynomial of degree $k$. Assume $A$ is an $n \times n$ constant invertible matrix. Then $u' = Au + f(t)c$ has a polynomial solution $u(t) = \sum_{j=0}^{k} c_j \frac{t^j}{j!}$ of degree $k$ with vector coefficients $\{c_j\}$ given by the relations

$$c_j = - \sum_{i=j}^{k} f^{(i)}(0) A^{j-i-1}c, \quad 0 \leq j \leq k.$$