

**The Integrating Factor Method  
for a  
Linear Differential Equation**  
 $y' + p(x)y = r(x)$

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## Superposition

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Consider the homogeneous equation

$$(1) \quad y' + p(x)y = 0$$

and the non-homogeneous equation

$$(2) \quad y' + p(x)y = r(x)$$

where  $p$  and  $r$  are continuous in an interval  $J$ .

### Theorem 1 (Superposition)

The general solution of the non-homogeneous equation (2) is given by

$$y = y_h + y_p$$

where  $y_h$  is the general solution of homogeneous equation (1) and  $y_p$  is a particular solution of non-homogeneous equation (2).

## Variation of Parameters

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The initial value problem

$$(3) \quad \mathbf{y}' + \mathbf{p}(x)\mathbf{y} = \mathbf{r}(x), \quad \mathbf{y}(x_0) = \mathbf{0},$$

where  $\mathbf{p}$  and  $\mathbf{r}$  are continuous in an interval containing  $x = x_0$ , has a particular solution

$$(4) \quad \mathbf{y}(x) = e^{-\int_{x_0}^x \mathbf{p}(s) ds} \int_{x_0}^x \mathbf{r}(t) e^{\int_{x_0}^t \mathbf{p}(s) ds} dt.$$

Formula (4) is called **variation of parameters**, for historical reasons.

The formula determines a particular solution  $\mathbf{y}_p$  which can be used in the superposition identity  $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$ .

While (4) has some appeal, applications use the **integrating factor method**, which is developed with indefinite integrals for computational efficiency. No one memorizes (4); they remember and study the *method*.

## Integrating Factor Identity

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The technique called the **integrating factor method** uses the replacement rule

$$(5) \quad \text{Fraction } \frac{(YW)'}{W} \text{ replaces } Y' + p(x)Y, \text{ where } W = e^{\int p(x)dx}.$$

The factor  $W = e^{\int p(x)dx}$  in (5) is called an **integrating factor**.

### Details

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Let  $W = e^{\int p(x)dx}$ . Then  $W' = pW$ , by the rule  $(e^x)' = e^x$ , the chain rule and the fundamental theorem of calculus  $(\int p(x)dx)' = p(x)$ .

Let's prove  $(WY)' / W = Y' + pY$ . The derivative product rule implies

$$\begin{aligned}(YW)' &= Y'W + YW' \\ &= Y'W + YpW \\ &= (Y' + pY)W.\end{aligned}$$

The proof is complete.

## The Integrating Factor Method

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**Standard Form** Rewrite  $y' = f(x, y)$  in the form  $y' + p(x)y = r(x)$  where  $p, r$  are continuous. The method applies only in case this is possible.

**Find  $W$**  Find a simplified formula for  $W = e^{\int p(x)dx}$ . The antiderivative  $\int p(x)dx$  can be chosen conveniently.

**Prepare for Quadrature** Obtain the new equation  $\frac{(yW)'}{W} = r$  by replacing the left side of  $y' + p(x)y = r(x)$  by equivalence (5).

**Method of Quadrature** Clear fractions to obtain  $(yW)' = rW$ . Apply the method of quadrature to get  $yW = \int r(x)W(x)dx + C$ . Divide by  $W$  to isolate the explicit solution  $y(x)$ .

Equation (5) is central to the method, because it collapses the two terms  $y' + py$  into a single term  $(yW)'/W$ ; the method of quadrature applies to  $(yW)' = rW$ . Literature calls the exponential factor  $W$  an **integrating factor** and equivalence (5) a **factorization** of  $Y' + p(x)Y$ .

## Integrating Factor Example

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**Example.** Solve the linear differential equation  $xy' + y = x^2$ .

**Solution:** The standard form of the linear equation is

$$y' + \frac{1}{x}y = x.$$

Let

$$W = e^{\int \frac{1}{x} dx} = x$$

and replace the LHS of the differential equation by  $(yW)'/W$  to obtain the quadrature equation

$$(yW)' = xW.$$

Apply quadrature to this equation, then divide by  $W$ , to give the answer

$$y = \frac{x^2}{3} + \frac{C}{x}.$$