

# Systems of Differential Equations

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## Characteristic Equation

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### Definition 1 (Characteristic Equation)

Given a square matrix  $A$ , the **characteristic equation** of  $A$  is the polynomial equation

$$\det(A - rI) = 0.$$

The determinant  $\det(A - rI)$  is formed by subtracting  $r$  from the diagonal of  $A$ .

The polynomial  $p(r) = \det(A - rI)$  is called the **characteristic polynomial**.

- If  $A$  is  $2 \times 2$ , then  $p(r)$  is a quadratic.
- If  $A$  is  $3 \times 3$ , then  $p(r)$  is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.

## Characteristic Equation Examples

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Create  $\det(A - rI)$  by subtracting  $r$  from the diagonal of  $A$ .

Evaluate by the cofactor rule.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 \\ 0 & 4 - r \end{vmatrix} = (2 - r)(4 - r)$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2 - r)(5 - r)(7 - r)$$

## Cayley-Hamilton

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### Theorem 1 (Cayley-Hamilton)

A square matrix  $A$  satisfies its own characteristic equation.

If  $p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \dots + a_0$ , then the result is the equation

$$(-A)^n + a_{n-1}(-A)^{n-1} + \dots + a_1(-A) + a_0I = 0,$$

where  $I$  is the  $n \times n$  identity matrix and  $0$  is the  $n \times n$  zero matrix.

## Cayley-Hamilton Example

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Assume

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$$

Then

$$p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2 - r)(5 - r)(7 - r)$$

and the Cayley-Hamilton Theorem says that

$$(2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

## Cayley-Hamilton-Ziebur Theorem

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### **Theorem 2 (Cayley-Hamilton-Ziebur Structure Theorem for $\vec{u}' = A\vec{u}$ )**

A component function  $u_k(t)$  of the vector solution  $\vec{u}(t)$  for  $\vec{u}'(t) = A\vec{u}(t)$  is a solution of the  $n$ th order linear homogeneous constant-coefficient differential equation whose characteristic equation is  $\det(A - rI) = 0$ .

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The theorem implies that the vector solution  $\vec{u}(t)$  of

$$\vec{u}' = A\vec{u}$$

is a vector linear combination of atoms constructed from the roots of the characteristic equation  $\det(A - rI) = 0$ .

## Proof of the Cayley-Hamilton-Ziebur Theorem

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Consider the case  $n = 2$ , because the proof details are similar in higher dimensions.

$$r^2 + a_1 r + a_0 = 0 \quad \text{Expanded characteristic equation}$$

$$A^2 + a_1 A + a_0 I = 0 \quad \text{Cayley-Hamilton matrix equation}$$

$$A^2 \vec{u} + a_1 A \vec{u} + a_0 \vec{u} = \vec{0} \quad \text{Right-multiply by } \vec{u} = \vec{u}(t)$$

$$\vec{u}'' = A \vec{u}' = A^2 \vec{u} \quad \text{Differentiate } \vec{u}' = A \vec{u}$$

$$\vec{u}'' + a_1 \vec{u}' + a_0 \vec{u} = \vec{0} \quad \text{Replace } A^2 \vec{u} \rightarrow \vec{u}'', A \vec{u} \rightarrow \vec{u}'$$

Then the components  $x(t)$ ,  $y(t)$  of  $\vec{u}(t)$  satisfy the two differential equations

$$\begin{aligned} x''(t) + a_1 x'(t) + a_0 x(t) &= 0, \\ y''(t) + a_1 y'(t) + a_0 y(t) &= 0. \end{aligned}$$

This system implies that the components of  $\vec{u}(t)$  are solutions of the second order DE with characteristic equation  $\det(A - rI) = 0$ .

## Cayley-Hamilton-Ziebur Method

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### The Cayley-Hamilton-Ziebur Method for $\vec{u}' = A\vec{u}$

Let  $\text{atom}_1, \dots, \text{atom}_n$  denote the atoms constructed from the  $n$ th order characteristic equation  $\det(A - rI) = 0$  by Euler's Theorem. The solution of

$$\vec{u}' = A\vec{u}$$

is given for some constant vectors  $\vec{d}_1, \dots, \vec{d}_n$  by the equation

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \dots + (\text{atom}_n)\vec{d}_n$$

## Cayley-Hamilton-Ziebur Method Conclusions

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- Solving  $\vec{u}' = A\vec{u}$  is reduced to finding the constant vectors  $\vec{d}_1, \dots, \vec{d}_n$ .
- The vectors  $\vec{d}_j$  are **not arbitrary**. They are **uniquely determined** by  $A$  and  $\vec{u}(0)$ !  
A general method to find them is to differentiate the equation

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \dots + (\text{atom}_n)\vec{d}_n$$

$n - 1$  times, then set  $t = 0$  and replace  $\vec{u}^{(k)}(0)$  by  $A^k\vec{u}(0)$  [because  $\vec{u}' = A\vec{u}$ ,  $\vec{u}'' = A\vec{u}' = AA\vec{u}$ , etc]. The resulting  $n$  equations in vector unknowns  $\vec{d}_1, \dots, \vec{d}_n$  can be solved by elimination.

- If all atoms constructed are base atoms constructed from real roots, then each  $\vec{d}_j$  is a constant multiple of a real eigenvector of  $A$ . Atom  $e^{rt}$  corresponds to the eigenpair equation  $A\mathbf{v} = r\mathbf{v}$ .

## A $2 \times 2$ Illustration

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Let's solve  $\vec{u}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{u}$ ,  $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ .

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The characteristic polynomial of the non-triangular matrix  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  is

$$\begin{vmatrix} 1 - r & 2 \\ 2 & 1 - r \end{vmatrix} = (1 - r)^2 - 4 = (r + 1)(r - 3).$$

Euler's theorem implies solution atoms are  $e^{-t}$ ,  $e^{3t}$ .

Then  $\vec{u}$  is a vector linear combination of the solution atoms,

$$\vec{u} = e^{-t}\vec{d}_1 + e^{3t}\vec{d}_2.$$

## How to Find $\vec{d}_1$ and $\vec{d}_2$

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We solve for vectors  $\vec{d}_1, \vec{d}_2$  in the equation

$$\vec{u} = e^{-t}\vec{d}_1 + e^{3t}\vec{d}_2.$$

**Advice:** Defined  $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ . Differentiate the above relation. Replace  $\vec{u}'$  via  $\vec{u}' = A\vec{u}$ , then set  $t = 0$  and replace  $\vec{u}(0)$  by  $\vec{d}_0$  in the two formulas to obtain the relations

$$\begin{aligned}\vec{d}_0 &= e^0\vec{d}_1 + e^0\vec{d}_2 \\ A\vec{d}_0 &= -e^0\vec{d}_1 + 3e^0\vec{d}_2\end{aligned}$$

We solve for  $\vec{d}_1, \vec{d}_2$  by elimination. Adding the equations gives  $\vec{d}_0 + A\vec{d}_0 = 4\vec{d}_2$  and then  $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  implies

$$\begin{aligned}\vec{d}_1 &= \frac{3}{4}\vec{d}_0 - \frac{1}{4}A\vec{d}_0 = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}, \\ \vec{d}_2 &= \frac{1}{4}\vec{d}_0 + \frac{1}{4}A\vec{d}_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.\end{aligned}$$

## Summary of the $2 \times 2$ Illustration

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The solution of the dynamical system

$$\vec{u}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{u}, \quad \mathbf{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

is a vector linear combination of solution atoms  $e^{-t}$ ,  $e^{3t}$  given by the equation

$$\vec{u} = e^{-t} \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

## A Matrix Method for Finding $\vec{d}_1$ and $\vec{d}_2$

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The Cayley-Hamilton-Ziebur Method produces a unique solution for  $\vec{d}_1, \vec{d}_2$  because the coefficient matrix

$$\begin{pmatrix} e^0 & e^0 \\ -e^0 & 3e^0 \end{pmatrix}$$

is exactly the Wronskian  $W$  of the basis of atoms  $e^{-t}, e^{3t}$  evaluated at  $t = 0$ . This same fact applies no matter the number of coefficients  $\vec{d}_1, \vec{d}_2, \dots$  to be determined.

The answer for  $\vec{d}_1$  and  $\vec{d}_2$  can be written in matrix form in terms of the transpose  $W^T$  of the Wronskian matrix as

$$\text{aug}(\vec{d}_1, \vec{d}_2) = \text{aug}(\vec{d}_0, A\vec{d}_0)(W^T)^{-1}.$$

## Solving a $2 \times 2$ Initial Value Problem by the Matrix Method

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$$\vec{u}' = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

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Then  $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ,  $A\vec{d}_0 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$  and

$$\text{aug}(\vec{d}_1, \vec{d}_2) = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}^T \right)^{-1} = \begin{pmatrix} -3/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix}.$$

The solution of the initial value problem is

$$\vec{u}(t) = e^{-t} \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \\ \frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix}.$$

## Other Representations of the Solution $\vec{u}$

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Let  $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$  be a solution basis for the  $n$ th order linear homogeneous constant-coefficient differential equation whose characteristic equation is  $\det(\mathbf{A} - r\mathbf{I}) = 0$ .

Consider the solution basis  $\mathbf{atom}_1, \mathbf{atom}_2, \dots, \mathbf{atom}_n$ . Each atom is a linear combination of  $\mathbf{y}_1, \dots, \mathbf{y}_n$ . Replacing the atoms in the formula

$$\vec{u}(t) = (\mathbf{atom}_1)\vec{d}_1 + \dots + (\mathbf{atom}_n)\vec{d}_n$$

by these linear combinations implies there are constant vectors  $\vec{d}_1, \dots, \vec{d}_n$  such that

$$\vec{u}(t) = \mathbf{y}_1(t)\vec{d}_1 + \dots + \mathbf{y}_n(t)\vec{d}_n$$

**Another General Solution of  $\vec{u}' = A\vec{u}$**

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**Theorem 3 (General Solution)**

The unique solution of  $\vec{u}' = A\vec{u}$ ,  $\vec{u}(0) = \vec{d}_0$  is

$$\mathbf{u}(t) = \phi_1(t)\mathbf{u}_0 + \phi_2(t)A\mathbf{u}_0 + \cdots + \phi_n(t)A^{n-1}\mathbf{u}_0$$

where  $\phi_1, \dots, \phi_n$  are linear combinations of atoms constructed from roots of the characteristic equation  $\det(A - rI) = 0$ , such that

$$\mathbf{Wronskian}(\phi_1(t), \dots, \phi_n(t))|_{t=0} = I.$$

## Proof of the theorem

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**Proof:** Details will be given for  $n = 3$ . The details for arbitrary matrix dimension  $n$  is a routine modification of this proof. The Wronskian condition implies  $\phi_1, \phi_2, \phi_3$  are independent. Then each atom constructed from the characteristic equation is a linear combination of  $\phi_1, \phi_2, \phi_3$ . It follows that the unique solution  $\vec{u}$  can be written for some vectors  $\vec{d}_1, \vec{d}_2, \vec{d}_3$  as

$$\vec{u}(t) = \phi_1(t)\vec{d}_1 + \phi_2(t)\vec{d}_2 + \phi_3(t)\vec{d}_3.$$

Differentiate this equation twice and then set  $t = 0$  in all 3 equations. The relations  $\vec{u}' = A\vec{u}$  and  $\vec{u}'' = A\vec{u}' = AA\vec{u}$  imply the 3 equations

$$\begin{aligned}\vec{d}_0 &= \phi_1(0)\vec{d}_1 + \phi_2(0)\vec{d}_2 + \phi_3(0)\vec{d}_3 \\ A\vec{d}_0 &= \phi_1'(0)\vec{d}_1 + \phi_2'(0)\vec{d}_2 + \phi_3'(0)\vec{d}_3 \\ A^2\vec{d}_0 &= \phi_1''(0)\vec{d}_1 + \phi_2''(0)\vec{d}_2 + \phi_3''(0)\vec{d}_3\end{aligned}$$

Because the Wronskian is the identity matrix  $I$ , then these equations reduce to

$$\begin{aligned}\vec{d}_0 &= 1\vec{d}_1 + 0\vec{d}_2 + 0\vec{d}_3 \\ A\vec{d}_0 &= 0\vec{d}_1 + 1\vec{d}_2 + 0\vec{d}_3 \\ A^2\vec{d}_0 &= 0\vec{d}_1 + 0\vec{d}_2 + 1\vec{d}_3\end{aligned}$$

which implies  $\vec{d}_1 = \vec{d}_0, \vec{d}_2 = A\vec{d}_0, \vec{d}_3 = A^2\vec{d}_0$ .

The claimed formula for  $\vec{u}(t)$  is established and the proof is complete.

## Change of Basis Equation

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Illustrated here is the change of basis formula for  $n = 3$ . The formula for general  $n$  is similar.

Let  $\phi_1(t)$ ,  $\phi_2(t)$ ,  $\phi_3(t)$  denote the linear combinations of atoms obtained from the vector formula

$$(\phi_1(t), \phi_2(t), \phi_3(t)) = (\text{atom}_1(t), \text{atom}_2(t), \text{atom}_3(t)) C^{-1}$$

where

$$C = \text{Wronskian}(\text{atom}_1, \text{atom}_2, \text{atom}_3)(0).$$

The solutions  $\phi_1(t)$ ,  $\phi_2(t)$ ,  $\phi_3(t)$  are called the **principal solutions** of the linear homogeneous constant-coefficient differential equation constructed from the characteristic equation  $\det(A - rI) = 0$ . They satisfy the initial conditions

$$\text{Wronskian}(\phi_1, \phi_2, \phi_3)(0) = I.$$