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Math 3150 Problems
Weeks 14-16
June 2009

Due date: See the internet due date. Problems are collected twice a week. Records are locked when the stack is returned. Records are only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Label each problem with its corresponding problem number, e.g., Prob3.1-4 or Xc1.2-4. Kindly label extra credit problems with label Extra Credit. You may attach this printed sheet to simplify your work.

Labeling. The label Probx.y-z means the problem is for chapter x, section y, problem z. When $y = 0$, then the problem does not have a textbook analog, it is a **background problem**. Otherwise, the problem number should match a corresponding problem in the textbook. The same labeling applies to extra credit problems, e.g., Xc1.0-4, Xc1.1-2.

Week 14: 7.1 – Fourier Integral Representation

Prob7.1-8a. (Fourier Integral Formulas)

Find functions $A(\omega)$, $B(\omega)$ for $f(x) = \begin{cases} 1, & 1 < |x| < 2, \\ 0 & \text{otherwise,} \end{cases}$ in the Fourier integral representation

$$f(x) = \int_0^{\infty} [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)] d\omega.$$

Display all details, including computer algebra steps, in evaluating the integrals

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt.$$

Prob7.1-8b. (Fourier Integral Convergence)

Given $f(x) = \begin{cases} 1, & 1 < |x| < 2, \\ 0 & \text{otherwise,} \end{cases}$, report the values of x for which $f(x)$ equals its Fourier integral.

Prob7.1-19. (Fourier Integral and Integration Formulas)

Invent a function $f(x)$ such that the Fourier Integral Representation implies the formula

$$e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos(\omega x)}{1 + \omega^2} d\omega.$$

Week 15: 7.2-7.3 – Fourier Transform

Prob7.2-20. (Fourier Transform)

Let $f(x) = x$ for $|x| < 1$ and $f(x) = 0$ elsewhere. Find the Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Prob7.2-18a. (Fourier Transform Rules)

State in detail the following operational rules for the Fourier Transform.

- (a) Linearity. (b) x -differentiation rule [parts]. (c) ω -differentiation rule. (d) Convolution rule.
 (e) x -axis shifting rule. (f) ω -axis shifting rule. (g) t -partial derivative rule.

Reference: Asmar's text, *PDE and BVP*, section 7.2, Theorems 1,2,3,4 and problems 7.2-19, 7.2-20. Section 7.3 for item (g). See also appendix B in Asmar.

Prob7.2-18b. (Fourier Transform Table)

Derive the table below, considered to be a suitable minimal table for using the Fourier transform on applied problems. In the table, we use the Fourier Transform notation

$$\mathcal{F}(f(x)) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx.$$

Define the **pulse function** by

$$\text{pulse}(x, a, b) = \begin{cases} 1 & a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{F}(\text{pulse}(x, a, b))(\omega)$	$= \frac{e^{i\omega a} - e^{-i\omega b}}{i\omega\sqrt{2\pi}}$
$\mathcal{F}\left(\frac{1}{a^2+x^2}\right)(\omega)$	$= \frac{\pi e^{-a \omega }}{a\sqrt{2\pi}} \quad (a > 0)$
$\mathcal{F}(e^{-a x })(\omega)$	$= \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \omega^2} \quad (a > 0)$
$\mathcal{F}(e^{-ax^2})(\omega)$	$= \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{2a}} \quad (a > 0)$

Reference: Asmar's text, *PDE and BVP*, appendix B.

Prob7.2-31a. (Fourier Transform Calculus)

Use the Fourier transform rules and Fourier transform table to derive a formula for $\mathcal{F}(f(x))$.

(a) $f(x) = \cos(x) e^{-x^2}$	(b) $f(x) = \sin(x) e^{- x }$	(c) $f(x) = \cos(x) \frac{1}{1+x^2}$
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Xc7.2-31a. (Fourier Transform Calculus)

Use the Fourier transform rules and Fourier transform table to derive a formula for $\mathcal{F}(f(x))$.

(a) $f(x) = \cos(2x) \frac{1}{4+x^2}$	(b) $f(x) = \cos(x) \text{pulse}(x, -1, 1)$	(c) $f(x) = \sin(x) \text{pulse}(x, -1, 1)$
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Prob7.2-45. (Fourier Transform Convolution)

- (a) Find the Fourier transform of the convolution of $xe^{-x^2/2}$ and e^{-x^2} .
 (b) Solve for $h(x)$ in the equation $\mathcal{F}(h(x)) = e^{-\omega^2} \frac{\sin \omega}{\omega}$, using the convolution theorem.

Xc7.2-47. (Fourier Transform Convolution)

Write a proof for the following convolution identities. The convolution of f and g is defined by the equation

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

- (a) $f * g = g * f$
 (b) $f * (g * h) = (f * g) * h$

Prob7.3-1. (Fourier Transform Method Wave Equation)

Solve the boundary value problem on $-\infty < x < \infty, t > 0$.

$$\begin{aligned}u_{tt}(x, t) &= u_{xx}(x, t), \\u(x, 0) &= \frac{1}{1+x^2}, \\u_t(x, 0) &= 0.\end{aligned}$$

Prob7.3-4. (Fourier Transform Method Heat Equation)

Solve the boundary value problem on $-\infty < x < \infty, t > 0$.

$$\begin{aligned}u_t(x, t) &= \frac{1}{100}u_{xx}(x, t), \\u(x, 0) &= 100 \mathbf{pulse}(x, -1, 1).\end{aligned}$$

Xc7.3-17. (Fourier Transform Method Infinite Beam)

Solve the boundary value problem on $-\infty < x < \infty, t > 0$.

$$\begin{aligned}u_{tt}(x, t) &= \frac{1}{100}u_{xxxx}(x, t), \\u(x, 0) &= 100 \mathbf{pulse}(x, -2, 2).\end{aligned}$$

Week 16: 7.4-7.5 – Heat Kernel and Poisson Integral Formula

Xc7.4-2. (Heat Kernel)

Solve the boundary value problem on $-\infty < x < \infty, t > 0$.

$$\begin{aligned}u_t(x, t) &= \frac{1}{100}u_{xx}(x, t), \\u(x, 0) &= 100 \mathbf{pulse}(x, -2, 0) + 50 \mathbf{pulse}(x, 0, 1).\end{aligned}$$

Xc7.4-6. (Heat Kernel)

Solve the boundary value problem on $-\infty < x < \infty, t > 0$.

$$\begin{aligned}u_t(x, t) &= \frac{1}{100}u_{xx}(x, t), \\u(x, 0) &= e^{-|x|}.\end{aligned}$$