Instructions: This exam is timed for 120 minutes. You will be given 30 extra minutes to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.
1. (Fourier Series)
   Let \( f(x) = -1 \) on the interval \( 0 < x < \pi \), \( f(x) = 1 \) on \( -\pi < x < 0 \). Let \( f(x) \) be defined at \( x = 0, \pi, -\pi \) to make its \( 2\pi \)-periodic extension \( g(x) \) satisfy \( g(-x) = g(x) \).

   (a) [25%] Display the formulas for the Fourier coefficients of \( f \), but do not calculate the integrals.

   (b) [25%] Compute the first two nonzero Fourier coefficients.

   (c) [25%] Let \( g(x) \) equal the Fourier series of \( f \). Calculate the value of \( g(x) \) for every \( x \) in \( -\infty < x < \infty \).

   (d) [25%] Draw a figure for this example, to illustrate Gibb’s phenomenon. Include detail about the Fourier series partial sum \( S_N(x) \), where \( N \) is the first index that includes the terms found in part (b) above.

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx \quad \text{Fourier series} \quad = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)
\]

\[
\langle g, 1 \rangle = a_0 \langle 1, 1 \rangle \implies a_0 = \frac{\langle g, 1 \rangle}{\langle 1, 1 \rangle} \implies a_0 = \frac{\int_{-\pi}^{\pi} g(x) \, dx}{2\pi}
\]

\[
\langle g, \cos nx \rangle = a_n \langle \cos nx, \cos nx \rangle \implies a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx) \, dx
\]

\[
\langle g, \sin nx \rangle = b_n \langle \sin nx, \sin nx \rangle \implies b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx) \, dx
\]

(b) \(-g(-x) = g(x)\) means \( g(x) \) is odd periodic. Then \( a_n = 0 \) for all \( n \).

\[
b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin x \, dx = -\frac{2}{\pi} \int_{0}^{\pi} \sin x \, dx = -\frac{2}{\pi} (-\cos x) \bigg|_{0}^{\pi} = -\frac{4}{\pi}
\]

\[
b_2 = -\frac{1}{\pi} \cdot 2 \cdot (1 - \cos 2x) \bigg|_{0}^{\pi} = 0
\]

\[
b_3 = -\frac{1}{\pi} \cdot 2 \cdot (1 - \cos 3x) \bigg|_{0}^{\pi} = -\frac{2}{\pi} \cdot \frac{1}{3}
\]

(c) \( \text{Value} = \frac{1}{2}[f(x+) + f(x-)] \)

\[
= \begin{cases} 
0 & -\pi < x < 0 \\
-1 & 0 < x < \pi \\
0 & x = \pi, -\pi
\end{cases}
\]

(d) \( S_N(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \sum_{n=1}^{\infty} b_n \sin(nx) \)

\[
S_0(x) = -\frac{4}{\pi} \sin(x) - \frac{4}{3\pi} \sin(3x)
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (CH3. Finite String: Fourier Series Solution)

Consider the problem

\[
\begin{aligned}
    u_t(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < 2\pi, \quad t > 0, \\
    u(0,t) &= 0, \quad t > 0, \\
    u(2\pi,t) &= 0, \quad t > 0, \\
    u(x,0) &= f(x), \quad 0 < x < 2\pi, \\
    u_t(x,0) &= 0, \quad 0 < x < 2\pi
\end{aligned}
\]

(a) [50%] Do the analysis of product solutions \( u = X(x)T(t) \).

(b) [25%] Display the series solution \( u(x,t) \). Include formulas for the coefficients.

(c) [25%] Find the solution \( u(x,t) \) when \( c = 1/\pi \) and \( f(x) = 7\sin(4x) - 8\sin(9x) \).

(a) \( u = X T \Rightarrow -\lambda = \frac{X''}{X} = \frac{T''}{c^2 T} \Rightarrow
\begin{align*}
    X'' + 2\pi \frac{X}{X} &= 0 \\
    X(0) &= X(2\pi) = 0 \\
    X &= \sin(\sqrt{\lambda} x) \\
    \lambda &= \frac{n^2}{\pi^2} \quad (n = 1, 2, 3, \ldots) \\
    T &= C_n \cos(\sqrt{\lambda} x) + D_n \sin(\sqrt{\lambda} x) \\
    \text{No sol for } \lambda = 0, \lambda \geq 0 \\
\end{align*}
\]

(b) \( u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(\pi x) + b_n \sin(\pi x)) \)

\[ \langle f, g \rangle = \int_0^{2\pi} fg \, dx \quad \langle f, \sin(n x) \rangle = a_n \langle \sin(\pi x), \sin(n x) \rangle \]

\[ a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(\pi x) \, dx, \quad b_n = 0 \]

(c) \( u(x,t) = 7 \sin(4x) \cos(4t/\pi) - 8 \sin(9x) \cos(9t/\pi) \)

By Fourier's method, since \( f(x) \) is a linear combination of eigenfunctions.
3. (CH4. Steady-State Heat Conduction on a Disk)

Consider the problem

\[
\begin{aligned}
&u_{rr}(r, \theta) + \frac{1}{r} u_r(r, \theta) + \frac{1}{r^2} u_{\theta\theta}(r, \theta) = 0, \quad 0 < r < 2, \quad 0 < \theta < 2\pi, \\
u(2, \theta) = f(\theta), \quad 0 < \theta < 2\pi.
\end{aligned}
\]

(a) [75%] A product solution has the form \( R(r) \Theta(\theta) \). Do the analysis to find formulas for \( R(r) \) and \( \Theta(\theta) \). Then display the series solution \( u(r, \theta) \).

(b) [25%] Find the coefficients in the series solution when \( f(\theta) = \theta \) on \( 0 < \theta < 2\pi \), \( f(0) = f(2\pi) = 0 \), \( f(\theta + 2\pi) = f(\theta) \).

(a) \( u = R \Theta \Rightarrow \frac{R''(r) + \frac{1}{r} R'(r) - \frac{n^2}{r^2} R(r)}{R} = -\frac{\Theta''}{\Theta} = \lambda \)

\[
\Rightarrow \Theta'' + \lambda \Theta = 0 \quad \text{with} \quad \Theta(0) \text{ periodic}, \quad \text{period} = 2\pi
\]

\[
\Rightarrow \Theta = c_1 \cos(\sqrt{\lambda} \theta) + c_2 \sin(\sqrt{\lambda} \theta)
\]

\[
\lambda = \lambda_0 \quad \text{possible}; \quad \lambda < 0 \text{ not allowed.}
\]

The \( R \)-equation is then \( r^2 R'' + r R' + (-n^2) R = 0 \), a Cauchy - Euler DE. A basis of solutions is \( r^n, \frac{1}{r^n} \). We exclude \( r^n \) because \( R(r) \) is defined at \( r = 0 \). Then \( R(r) = r^n \).

\[
u(r, \theta) = \sum_{n=1}^{\infty} r^n \left( a_n \cos(n \theta) + b_n \sin(n \theta) \right) + a_0
\]

(b) \( f(\theta) = \sum_{n=1}^{\infty} 2^n a_n \cos(n \theta) + 2^n b_n \sin(n \theta) + a_0 \)

\[
\int_0^{2\pi} \Theta \cos(n \theta) d\theta = 2^n a_n \int_0^{2\pi} \cos^n(\theta) d\theta = \pi 2^n a_n
\]

\[
\int_0^{2\pi} \Theta \sin(n \theta) d\theta = 2^n b_n \int_0^{2\pi} \sin^n(\theta) d\theta = \pi 2^n b_n
\]

Formulas

\[
\frac{d}{d\theta} (\Theta \cos(n \theta)) = \cos(n \theta) - n \Theta \sin(n \theta)
\]

\[
\frac{d}{d\theta} (\Theta \sin(n \theta)) = \sin(n \theta) + n \Theta \cos(n \theta)
\]

Imply

\[
\begin{cases}
\pi 2^n a_n = 0 \\
\pi 2^n b_n = \frac{2\pi}{n}
\end{cases}
\]

\[
\Rightarrow \begin{cases}
a_n = 0 \\
b_n = \frac{2^n}{2n} \left( \frac{2\pi}{n} \right) \\
a_0 = 2\pi^2
\end{cases}
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
4. (CH4. Poisson Problem)
Solve for \( u(x, y) \) in the Poisson problem

\[
\begin{aligned}
\begin{cases}
  u_{xx} + u_{yy} &= \sin(\pi x) \sin(\pi y), & 0 < x < 1, \ 0 < y < 1, \\
  u(x, 0) &= \sin(\pi x) + 3 \sin(4\pi x), & 0 < x < 1, \\
  u(x, y) &= 0 & \text{on the other 3 boundary edges.}
\end{cases}
\end{aligned}
\]

Then \( u = u_1 + u_2 \), where \( u_1 \) solves (1) and \( u_2 \) solves (2)

(1) Eigenfunctions \( \Delta = \partial_{xx} + \partial_{yy} \) are \( \phi_{mn} = \sin(m\pi x) \sin(n\pi y) \), so \( \Delta \phi_{mn} = -\lambda_{mn} \phi_{mn} \) and \( \lambda_{mn} = (m\pi)^2 + (n\pi)^2 \). Then

\[
\begin{aligned}
  u_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \phi_{mn} \\
  \Delta u_1 &= \sin(\pi x) \sin(\pi y) = \phi_{11} \\
  \Rightarrow \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} (-\lambda_{mn}) \phi_{mn} &= \phi_{11} \\
  \Rightarrow \quad E_{11} (-\lambda_{11}) &= 1 \quad \text{and all other } E_{mn} = 0 \\
  \Rightarrow \quad u_1 &= \frac{-\phi_{11}}{\lambda_{11}} = \frac{-\sin(\pi x) \sin(\pi y)}{2\pi^2}
\end{aligned}
\]

(2) Separate variables \( u = X(x) Y(y) \) in (2) to get \( \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \)

Then \( X'' - \lambda X = 0 \), \( X(0) = X(1) = 0 \) \( \Rightarrow Y = c_1 \sinh(\sqrt{\lambda} x) \). The series solution is

\[
\begin{aligned}
  u_2 &= \sum_{n=1}^{\infty} b_n \sin(n\pi x) \sinh(n\pi(1-y)) \\
  u_2 &= \sin(\pi x) \frac{\sinh(\pi(1-y))}{\sinh(\pi)} + 3 \sin(4\pi x) \frac{\sinh(4\pi(1-y))}{\sinh(4\pi)} 
\end{aligned}
\]

by Fourier's method.

Use this page to start your solution. Attach extra pages as needed, then staple.
5. (CH4. Heat Equation)
Solve the insulated rod heat conduction problem

\[
\begin{align*}
  u_t(x,t) &= u_{xx}(x,t), & 0 < x < 1, & t > 0, \\
  u(0,t) &= 0, & t > 0, \\
  u(1,t) &= 0, & t > 0, \\
  u(x,0) &= 1 & 0 < x < 1.
\end{align*}
\]

Separate variables \( u = X(x) T(t) \) to get \( \frac{X''}{X} = \frac{T'}{T} = \lambda \)
with \( \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases} \) and \( T' - \lambda T = 0 \). Then \( X(x) = \sin(\sqrt{\lambda} x) \)
with \( \sqrt{\lambda} \cdot 1 = n \pi \). The second solution is

\[
X(x) = \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 t} \sin(n \pi x)
\]

\[
f(x) = \sum_{n=1}^{\infty} c_n e^{0} \sin(n \pi x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}
\]

\[
< f, g > = \int_{-1}^{1} f(x) g(x) dx
\]
selected because we use half-range exp.

\[
< f, \sin(n \pi x) > = \sum_{n=1}^{\infty} c_n < \sin(k \pi x), \sin(n \pi x) >
\]

The system \( \{ \sin(k \pi x) \}_{k=1}^{\infty} \) is orthonormal on \([-1,1]\).

\[
\Rightarrow \int_{-1}^{1} \sin(n \pi x) dx = c_n \int_{-1}^{1} \sin^2(n \pi x) dx
\]

\[
\Rightarrow 2 \int_{0}^{1} \sin(n \pi x) dx = c_n \cdot 1
\]

\[
\Rightarrow c_n = \int_{0}^{1} \frac{1}{n \pi} \sin(n \pi x) dx
\]

\[
\Rightarrow n \text{ even}
\]

\[
\Rightarrow n \text{ odd}
\]

\[
U(x,t) = \frac{4}{\pi} \sum_{n \text{ odd}} e^{-n^2 \pi^2 t} \frac{\sin(n \pi x)}{n}
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
6. (CH7. Fourier Transform: Infinite Rod)
Let \( f(x) = 1 \) for \( 0 < x < 2 \) and \( f(x) = 0 \) elsewhere on \( -\infty < x < \infty \). Solve the insulated rod heat conduction problem

\[
\begin{align*}
    u_t(x,t) &= u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0, \\
    u(x,0) &= f(x), \quad -\infty < x < \infty.
\end{align*}
\]

The Heat Kernel Theorem will be used to obtain the solution.

\[
    u(x,t) = \text{convolution of } f \text{ and } \text{heat kernel}
\]

\[
    u(x,t) = \frac{1}{c\sqrt{2\pi t}} e^{-\frac{x^2}{4c^2 t}} * f(x)
\]

\[
    = \frac{1}{c\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(u) e^{-\frac{(x-u)^2}{4c^2 t}} \, du
\]

\[
    = \frac{1}{c\sqrt{\pi t}} \int_{0}^{2} 1 \cdot e^{-\frac{(x-u)^2}{4c^2 t}} \, du
\]

This can be written in terms of \( \text{erf}(x) \), but that is an optional step, not required.

Use this page to start your solution. Attach extra pages as needed, then staple.