Partial Differential Equations 3150
Final Exam
Fall 2009

Instructions: This exam is timed for 120 minutes. You will be given 30 extra minutes to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

Keep this page for your records, or discard it.

Name KEY

3150 Exam 2 F2009

Typo: $g(-x) = -g(x)$

1. (Fourier Series)

Let $f(x) = -1$ on the interval $0 < x < \pi$, $f(x) = 1$ on $-\pi < x < 0$. Let $f(x)$ be defined at $x = 0, \pi, -\pi$ to make its 2π-periodic extension $g(x)$ satisfy $g(-x) = -g(x)$. minus sign added at exam time

(a) [25%] Display the formulas for the Fourier coefficients of f, but do not calculate the integrals.

(b) [25%] Compute the first two nonzero Fourier coefficients.

(c) [25%] Let $g(x)$ equal the Fourier series of f. Calculate the value of $g(x)$ for every x in $-\infty < x < \infty$.

(d) [25%] Draw a figure for this example, to illustrate Gibb's phenomenon. Include detail about the Fourier series partial sum $S_N(x)$, where N is the first index that includes the terms found in part (b) above.

(a) $\langle f,g\rangle = \int_{-\pi}^{\pi} fg\,dx$ Fourier series $= a_0 + \sum_1^\infty a_n \cos(nx) + b_n \sin(nx)$

$\langle g,1\rangle = a_0\langle 1,1\rangle \Rightarrow a_0 = \dfrac{\langle g,1\rangle}{\langle 1,1\rangle} \Rightarrow a_0 = \dfrac{\int_{-\pi}^{\pi} g\,dx}{2\pi}$

$\langle g, \cos nx\rangle = a_n \langle \cos nx, \cos nx\rangle \Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx$

$\langle g, \sin nx\rangle = b_n \langle \sin nx, \sin nx\rangle \Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)\,dx$

(b) $-g(-x) = g(x)$ means $g(x)$ is odd periodic. Then $a_n = 0$ for all n.

$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin x\,dx = \frac{-2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{-2}{\pi}(1-\cos x)\Big|_0^{\pi} = \frac{-4}{\pi}$

$b_2 = \frac{-1}{\pi}\cdot 2\cdot(1-\cos 2x)\frac{1}{2}\Big|_0^{\pi} = 0$

$b_3 = \frac{-1}{\pi}\cdot 2\cdot(1-\cos 3x)\frac{1}{3}\Big|_0^{\pi} = \frac{-4}{\pi}\cdot\frac{1}{3}$

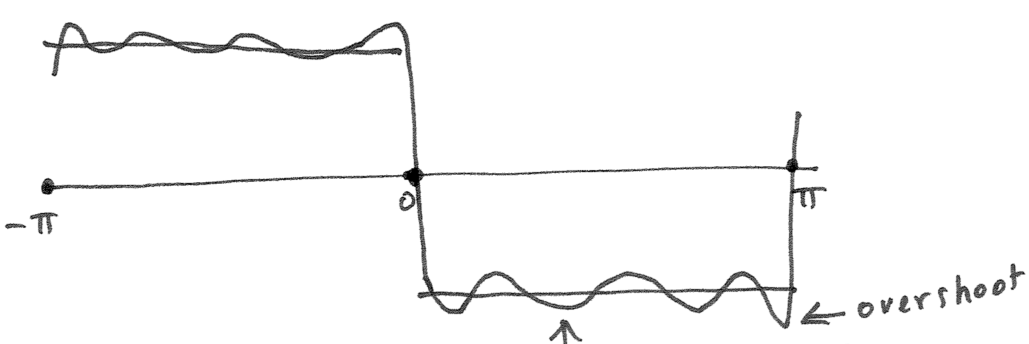
$\boxed{b_1 = \frac{-4}{\pi},\ b_3 = \frac{-4}{3\pi}}$

(c) value $= \frac{1}{2}(f(x+) + f(x-))$

$= \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 < x < \pi \\ 0 & x = 0, \pi, -\pi \end{cases}$

(d) $S_N(x) = a_0 + \sum_1^\infty a_n\cos(nx) + b_n\sin(nx) = \sum_1^\infty b_n \sin(nx)$

$S_3(x) = \frac{-4}{\pi}\sin(x) - \frac{4}{3\pi}\sin(3x)$ vanishes at $x = 0, \pi, -\pi$



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2. (CH3. Finite String: Fourier Series Solution)
Consider the problem

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & 0 < x < 2\pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(2\pi,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < 2\pi, \\ u_t(x,0) = 0, & 0 < x < 2\pi \end{cases}$$

(a) [50%] Do the analysis of product solutions $u = X(x)T(t)$.

(b) [25%] Display the series solution $u(x,t)$. Include formulas for the coefficients.

(c) [25%] Find the solution $u(x,t)$ when $c = 1/\pi$ and $f(x) = 7\sin(4x) - 8\sin(9x)$.

(a) $u = XT \Rightarrow -\lambda = \frac{X''}{X} = \frac{T''}{c^2 T} \Rightarrow$ $X'' + \lambda X = 0$, $X(0) = X(2\pi) = 0$, $X = \sin(\sqrt{\lambda}x)$, No sol for $\lambda = 0, \lambda < 0$, $2\sqrt{\lambda}\pi = n\pi \Rightarrow \boxed{\lambda_n = \frac{n^2}{4}}$ and $T'' + \lambda c^2 T = 0$, $T = c_1\cos(\sqrt{\lambda}ct) + c_2\sin(\sqrt{\lambda}ct)$

(b) $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{nx}{2}\right)\left(a_n\cos\left(\frac{nct}{2}\right) + b_n\sin\left(\frac{nct}{2}\right)\right)$

$\langle f,g\rangle = \int_0^{2\pi} fg\,dx.$ $\quad \langle f, \sin(\frac{nx}{2})\rangle = a_n\langle \sin(\frac{nx}{2}), \sin(\frac{nx}{2})\rangle$

$\langle 0, \sin(\frac{nx}{2})\rangle = b_n\left(\frac{nc}{2}\right)\langle \sin(\frac{nx}{2}), \sin(\frac{nx}{2})\rangle$

$a_n = \dfrac{\int_0^{2\pi} f(x)\sin\left(\frac{nx}{2}\right)dx}{\pi}$, $b_n = 0$

(c) $u(x,t) = 7\sin(4x)\cos(4t/\pi) - 8\sin(9x)\cos(9t/\pi)$

by Fourier's method, since $f(x)$ is a linear combination of eigenfunctions.

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3. (CH4. Steady-State Heat Conduction on a Disk)
Consider the problem

$$\begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 0 < r < 2, \quad 0 < \theta < 2\pi, \\ u(2,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

(a) [75%] A product solution has the form $R(r)\Theta(\theta)$. Do the analysis to find formulas for $R(r)$ and $\Theta(\theta)$. Then display the series solution $u(r,\theta)$.

(b) [25%] Find the coefficients in the series solution when $f(\theta) = \theta$ on $0 < \theta < 2\pi$, $f(0) = f(2\pi) = 0$, $f(\theta + 2\pi) = f(\theta)$.

(a) $u = R\Theta \Rightarrow \dfrac{r^2(R'' + \frac{1}{r}R')}{R} = -\dfrac{\Theta''}{\Theta} = \lambda$

$\Rightarrow \Theta'' + \lambda\Theta = 0$ with $\Theta(\theta)$ periodic, period $= 2\pi$

$\Rightarrow \Theta = c_1 \cos(\sqrt{\lambda}\,\theta) + c_2 \sin(\sqrt{\lambda}\,\theta)$

and $\boxed{\lambda_n = n^2}$ $\lambda = 0$ possible; $\lambda < 0$ not allowed.

The R-equation is then $r^2R'' + rR' + (-n^2)R = 0$, a Cauchy-Euler DE. A basis of solutions is r^n, r^{-n}. We exclude r^{-n} because $R(r)$ is defined at $r = 0$. Then $R(r) = r^n$.

$$u(r,\theta) = \sum_{n=1}^{\infty} r^n\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) + a_0$$

(b) $$f(\theta) = \sum_{n=1}^{\infty} 2^n a_n \cos(n\theta) + 2^n b_n \sin(n\theta) + a_0$$

$$\int_0^{2\pi} \theta\cos(n\theta)\,d\theta = 2^n a_n \int_0^{2\pi} \cos^2(n\theta)\,d\theta = \pi 2^n a_n$$

$$\int_0^{2\pi} \theta\sin(n\theta)\,d\theta = 2^n b_n \int_0^{2\pi} \sin^2(n\theta)\,d\theta = \pi 2^n b_n$$

Formulas $\dfrac{d}{d\theta}(\theta\cos(n\theta)) = \cos(n\theta) - n\theta\sin(n\theta)$

$\dfrac{d}{d\theta}(\theta\sin(n\theta)) = \sin(n\theta) + n\theta\cos(n\theta)$

imply

$$\begin{cases} \pi 2^n a_n = 0 \\ \pi 2^n b_n = \frac{2\pi}{n} \end{cases} \Rightarrow \begin{cases} a_n = 0 \\ b_n = 2^{-n}\left(\frac{2}{n}\right) \\ a_0 = 2\pi^2 \end{cases}$$

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4. (CH4. Poisson Problem) 3

Solve for $u(x, y)$ in the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} &= \sin(\pi x)\sin(\pi y), & 0 < x < 1, \quad 0 < y < 1, \\ u(x,0) &= \sin(\pi x) + 3\sin(4\pi x), & 0 < x < 1, \\ u(x,y) &= 0 & \text{on the other 3 boundary edges.} \end{cases}$$

(1) $\begin{cases} u_{xx} + u_{yy} = \sin(\pi x)\sin(\pi y) \\ u(x,y) = 0 \text{ on all 4 edges} \end{cases}$ (2) $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = \sin(\pi x) + 3\sin(4\pi x) \\ u(x,y) = 0 \text{ on other 3 edges} \end{cases}$

Then $u = u_1 + u_2$, where u_1 solves (1) and u_2 solves (2)

(1) Eigenfunctions of $\Delta = \partial_{xx} + \partial_{yy}$ are $\phi_{mn} = \sin(m\pi x)\sin(n\pi y)$, so that $\Delta\phi_{mn} = -\lambda_{mn}\phi_{mn}$ and $\lambda_{mn} = (m\pi)^2 + (n\pi)^2$. Then

$$u_1 = \sum_1^\infty \sum_1^\infty E_{mn}\phi_{mn}$$

Then

$$\Delta u_1 = \sin(\pi x)\sin(\pi y) = \phi_{11}$$

$$\Rightarrow \sum_1^\infty \sum_1^\infty E_{mn}(-\lambda_{mn})\phi_{mn} = \phi_{11}$$

$$\Rightarrow E_{11}(-\lambda_{11}) = 1 \text{ and all other } E_{mn} = 0$$

$$\Rightarrow u_1 = \frac{-\phi_{11}}{\lambda_{11}} = -\frac{\sin(\pi x)\sin(\pi y)}{2\pi^2}$$

(2) Separate variables $u = XY$ in (2) to get $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases}$ and $X(x) = \sin(\sqrt{\lambda}\, x)$, $\sqrt{\lambda} = n\pi$

Then $Y'' - \lambda Y = 0$, $Y(1) = 0 \Rightarrow y = c_1 \sinh(\sqrt{\lambda}(1-y))$. The series solution is

$$u_2 = \sum_1^\infty b_n \sin(n\pi x)\sinh(n\pi(1-y))$$

$$u_2 = \sin(\pi x)\frac{\sinh(\pi(1-y))}{\sinh(\pi)} + 3\sin(4\pi x)\frac{\sinh(4\pi(1-y))}{\sinh(4\pi)}$$

by Fourier's method

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5. (CH4. Heat Equation)
Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t), & 0<x<1, & t>0, \\ u(0,t) &= 0, & & t>0, \\ u(1,t) &= 0, & & t>0, \\ u(x,0) &= 1 & 0<x<1. & \end{cases}$$

Separate variables $u = X(x)\,T(t)$ to get $\frac{X''}{X} = -\frac{T'}{T} = -\lambda$

with $\begin{cases} X''+\lambda X = 0 \\ X(0)=X(1)=0 \end{cases}$ and $\begin{cases} T'-\lambda T=0 \\ T(0)\neq 0 \end{cases}$. Then $X(x) = \sin(\sqrt{\lambda}\,x)$, $T(t) = e^{-\lambda t}\cdot c_1$

with $\sqrt{\lambda}\cdot 1 = n\pi$. The series solution is

$$u(x,t) = \sum_1^\infty c_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$f(x) = \sum_1^\infty c_n e^0 \sin(n\pi x) = \begin{cases} 1 & 0<x<1 \\ -1 & -1<x<0 \end{cases}$$

$\langle f,g\rangle = \int_{-1}^1 fg\,dx$ selected because we use half-range exp.

$$\langle f, \sin(n\pi x)\rangle = \sum_{k=1}^\infty c_n \langle \sin(k\pi x), \sin(n\pi x)\rangle$$

The system $\{\sin(k\pi x)\}_{k=1}^\infty$ is orthogonal on $[-1,1]$.

$$\Rightarrow \int_{-1}^1 f(x)\sin n\pi x\,dx = c_n\int_{-1}^1 \sin^2(n\pi x)\,dx$$

$$\Rightarrow 2\int_0^1 \sin(n\pi x)\,dx = c_n\cdot 1$$

$$\Rightarrow c_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$u(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}} e^{-n^2\pi^2 t}\,\frac{\sin(n\pi x)}{n}$$

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6. (CH7. Fourier Transform: Infinite Rod)
Let $f(x) = 1$ for $0 < x < 2$ and $f(x) = 0$ elsewhere on $-\infty < x < \infty$. Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases}$$

The Heat Kernel Theorem will be used to obtain the solution.

$u(x,t) =$ convolution of f and the heat kernel

$$u(x,t) = \frac{1}{c\sqrt{2t}} e^{\frac{-x^2}{4c^2t}} * f(x)$$

$$= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(u) e^{-\frac{(x-u)^2}{4c^2t}} du$$

$$= \frac{1}{2c\sqrt{\pi t}} \int_0^2 1 \cdot e^{-\frac{(x-u)^2}{4c^2t}} du$$

This can be written in terms of erf(x), but that is an optional step, not required.

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