Chapter 2

First Order Differential Equations

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The subject of the chapter is the first order differential equation $y' = f(x, y)$. The study includes closed-form solution formulas for special equations, numerical solutions and some applications to science and engineering.

2.1 The Method of Quadrature

The **method of quadrature** refers to the technique of integrating both sides of an equation, hoping thereby to extract a solution formula. The name *quadrature* originates in geometry, where *quadrature* means *finding area*, a task overtaken in modern mathematics by *integration.* The naming convention is obeyed by `maple`, which lists it as its first method for solving differential equations. Below, Theorem 1, proved on page 71, isolates the requirements which make this method successful.

Theorem 1 (Quadrature)
Let $F(x)$ be continuous on $a < x < b$. Assume $a < x_0 < b$ and $-\infty < y_0 < \infty$. Then the initial value problem

$$y' = F(x), \quad y(x_0) = y_0 \tag{1}$$

has the unique solution

$$y(x) = y_0 + \int_{x_0}^{x} F(t)dt. \tag{2}$$

To apply the *method of quadrature* means: (i) Calculate a candidate solution formula by the *working rule* below; (ii) Verify the solution.

> To solve $y' = f(x, y)$ when f is independent of y, integrate on variable x across the equation.

River Crossing

A boat crosses a river at fixed speed with power applied perpendicular to the shoreline. Is it possible to estimate the boat's downstream location? The answer is *yes*. The problem's variables are

x	Distance from shore,	w	Width of the river,
y	Distance downstream,	v_b	Boat velocity (dx/dt),
t	Time in hours,	v_r	River velocity (dy/dt).

The calculus chain rule $dy/dx = (dy/dt)/(dx/dt)$ is applied, using the symbols v_r and v_b instead of dy/dt and dx/dt, to give the *model equation*

$$\frac{dy}{dx} = \frac{v_r}{v_b}. \tag{3}$$

Stream Velocity. The downstream river velocity will be approximated by $v_r = kx(w - x)$, where $k > 0$ is a constant. This equation gives velocity $v_r = 0$ at the two shores $x = 0$ and $x = w$, while the **maximum stream velocity** at the center $x = w/2$ is (see page 71)

$$v_c = \frac{kw^2}{4}. \tag{4}$$

Special River-Crossing Model. The model equation (3) using $v_r = kx(w-x)$ and the constant k defined by (4) give the initial value problem

$$\frac{dy}{dx} = \frac{4v_c}{v_b w^2}x(w - x), \quad y(0) = 0. \tag{5}$$

The solution of (5) by the method of quadrature is

$$y = \frac{4v_c}{v_b w^2}\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right), \tag{6}$$

where w is the river's width, v_c is the river's midstream velocity and v_b is the boat's velocity. In particular, the boat's **downstream drift** on the opposite shore is $\frac{2}{3}w(v_c/v_b)$. See *Technical Details* page 71.

Examples

1 **Example (Quadrature)** Solve $y' = 3e^x$, $y(0) = 0$.

Solution:

Candidate solution. The *working rule* is applied.

$y'(t) = 3e^t$	Copy the equation, x replaced by t.
$\int_0^x y'(t)dt = \int_0^x 3e^t dt$	Integrate across $0 \le t \le x$.
$y(x) - y(0) = 3e^x - 3$	Fundamental theorem of calculus, page 682.
$y(x) = 3e^x - 3$	Candidate solution found. Used $y(0) = 0$.

Verify solution. Let $y = 3e^x - 3$. The initial condition $y(0) = 0$ follows from $e^0 = 1$. To verify the differential equation, the steps are:

$\text{LHS} = y'$	Left side of the differential equation.
$= (3e^x - 3)'$	Substitute the expression for y.
$= 3e^x - 0$	Sum rule, constant rule and $(e^u)' = u'e^u$.
$= \text{RHS}$	Solution verified.

2 **Example (River Crossing)** A boat crosses a mile-wide river at 3 miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is 10 miles per hour. Find the transit time and the downstream drift to the opposite shore.

Solution: The answers, justified below, are 20 minutes and 20/9 miles.

Transit time. This is the time it takes to reach the opposite shore. The layman answer of 20 minutes is correct, because the boat goes 3 miles in one hour, hence 1 mile in 1/3 of an hour, perpendicular to the shoreline.

Downstream drift. This is the value $y(1)$, where y is the solution of equation (5), with $v_c = 10$, $v_b = 3$, $w = 1$, all distances in miles. The special model is

$$\frac{dy}{dx} = \frac{40}{3}x(1-x), \quad y(0) = 0.$$

The solution given by equation (6) is $y = \frac{40}{3}\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2\right)$ and the downstream drift is then $y(1) = 20/9$ miles. This answer is 2/3 of the layman's answer of (1/3)(10) miles; the explanation is that the boat is pushed downstream at a variable rate from 0 to 10 miles per hour.

Details and Proofs

Proof of Theorem 1:

Uniqueness. Let $y(x)$ be any solution of (1). It will be shown that $y(x)$ is given by the solution formula (2).

$y(x) = y(0) + \int_{x_0}^{x} y'(t)dt$	Fundamental theorem of calculus, page 682.
$= y_0 + \int_{x_0}^{x} F(t)dt$	Use (1).

Verification of the Solution. Let $y(x)$ be given by solution formula (2). It will be shown that $y(x)$ solves initial value problem (1).

$y'(x) = \left(y_0 + \int_{x_0}^{x} F(t)dt\right)'$	Compute the derivative from (2).
$= F(x)$	Apply the fundamental theorem of calculus.

The initial condition is verified in a similar manner:

$y(x_0) = y_0 + \int_{x_0}^{x_0} F(t)dt$	Apply (2) with $x = x_0$.
$= y_0$	The integral is zero: $\int_a^a F(x)dx = 0$.

The proof is complete.

Technical Details for (4): The maximum of a continuously differentiable function $f(x)$ on $0 \le x \le w$ can be found by locating the critical points (i.e., where $f'(x) = 0$) and then testing also the endpoints $x = 0$ and $x = w$. The derivative $f'(x) = k(w - 2x)$ is zero at $x = w/2$. Then $f(w/2) = kw^2/4$. This value is the maximum of f, because $f = 0$ at the endpoints.

Technical Details for (6): Let $a = \dfrac{4v_c}{v_b w^2}$. Then

$y = y(0) + \int_0^x y'(t)dt$	Method of quadrature.
$= 0 + a\int_0^x t(w-t)dt$	By (5), $y' = at(w-t)$.
$= a\left(-\frac{1}{3}x^3 + \frac{1}{2}wx^2\right)$.	Integral table.

To compute the downstream drift, evaluate $y(w) = a\dfrac{w^3}{6}$ or $y(w) = \dfrac{2w}{3}\dfrac{v_c}{v_b}$.

Exercises 2.1

Quadrature. Find a candidate solution for each initial value problem and verify the solution. See Example 1, page 70.

1. $y' = 4e^{2x}$, $y(0) = 0$.

2. $y' = 2e^{4x}$, $y(0) = 0$.

3. $(1 + x)y' = x$, $y(0) = 0$.

4. $(1 - x)y' = x$, $y(0) = 0$.

5. $y' = \sin 2x$, $y(0) = 1$.

6. $y' = \cos 2x$, $y(0) = 1$.

7. $y' = xe^x$, $y(0) = 0$.

8. $y' = xe^{-x^2}$, $y(0) = 0$.

9. $y' = \tan x$, $y(0) = 0$.

10. $y' = 1 + \tan^2 x$, $y(0) = 0$.

11. $(1+x^2)y' = 1$, $y(0) = 0$.

12. $(1+4x^2)y' = 1$, $y(0) = 0$.

13. $y' = \sin^3 x$, $y(0) = 0$.

14. $y' = \cos^3 x$, $y(0) = 0$.

15. $(1+x)y' = 1$, $y(0) = 0$.

16. $(2+x)y' = 2$, $y(0) = 0$.

17. $(2+x)(1+x)y' = 2$, $y(0) = 0$.

18. $(2+x)(3+x)y' = 3$, $y(0) = 0$.

19. $y' = \sin x \cos 2x$, $y(0) = 0$.

20. $y' = (1+\cos 2x)\sin 2x$, $y(0) = 0$.

River Crossing. A boat crosses a river of width w miles at v_b miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is v_c miles per hour. Find the transit time and the downstream drift to the opposite shore. See Example 2, page 70, and the details for (6).

21. $w = 1$, $v_b = 4$, $v_c = 12$

22. $w = 1$, $v_b = 5$, $v_c = 15$

23. $w = 1.2$, $v_b = 3$, $v_c = 13$

24. $w = 1.2$, $v_b = 5$, $v_c = 9$

25. $w = 1.5$, $v_b = 7$, $v_c = 16$

26. $w = 2$, $v_b = 7$, $v_c = 10$

27. $w = 1.6$, $v_b = 4.5$, $v_c = 14.7$

28. $w = 1.6$, $v_b = 5.5$, $v_c = 17$

Fundamental Theorem I. Verify the identity. Use the fundamental theorem of calculus part (b), page 682.

29. $\int_0^x (1+t)^3 dt = \frac{1}{4}\left((1+x)^4 - 1\right)$.

30. $\int_0^x (1+t)^4 dt = \frac{1}{5}\left((1+x)^5 - 1\right)$.

31. $\int_0^x te^{-t} dt = -xe^{-x} - e^{-x} + 1$.

32. $\int_0^x te^{t} dt = xe^{x} - e^{x} + 1$.

Fundamental Theorem II. Differentiate. Use the fundamental theorem of calculus part (b), page 682.

33. $\int_0^{2x} t^2 \tan(t^3) dt$.

34. $\int_0^{3x} t^3 \tan(t^2) dt$.

35. $\int_0^{\sin x} te^{t+t^2} dt$.

36. $\int_0^{\sin x} \ln(1+t^3) dt$.

Fundamental Theorem III. Integrate $\int_0^1 f(x)dx$. Use the fundamental theorem of calculus part (a), page 682. Check answers with computer or calculator assist. Some require a clever u-substitution or an integral table.

37. $f(x) = x(x-1)$

38. $f(x) = x^2(x+1)$

39. $f(x) = \cos(3\pi x/4)$

40. $f(x) = \sin(5\pi x/6)$

41. $f(x) = \dfrac{1}{1+x^2}$

42. $f(x) = \dfrac{2x}{1+x^4})$

43. $f(x) = x^2 e^{x^3}$

44. $f(x) = x(\sin(x^2) + e^{x^2})$

45. $f(x) = \dfrac{1}{\sqrt{-1+x^2}}$

46. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

47. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

48. $f(x) = \dfrac{1}{\sqrt{1+4x^2}}$

49. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

50. $f(x) = \dfrac{4x}{\sqrt{1-4x^2}}$

51. $f(x) = \dfrac{\cos x}{\sin x}$

52. $f(x) = \dfrac{\cos x}{\sin^3 x}$

53. $f(x) = \dfrac{e^x}{1+e^x}$

54. $f(x) = \dfrac{\ln|x|}{x}$

55. $f(x) = \sec^2 x$

56. $f(x) = \sec^2 x - \tan^2 x$

57. $f(x) = \csc^2 x$

58. $f(x) = \csc^2 x - \cot^2 x$

59. $f(x) = \csc x \cot x x$

60. $f(x) = \sec x \tan x x$

Integration by Parts. Integrate $\int_0^1 f(x)dx$ by parts, $\int u dv = uv - \int v du$. Check answers with computer or calculator assist.

61. $f(x) = xe^x$

62. $f(x) = xe^{-x}$

63. $f(x) = \ln|x|$

64. $f(x) = x\ln|x|$

65. $f(x) = x^2 e^{2x}$

66. $f(x) = (1+2x)e^{2x}$

67. $f(x) = x\cosh x$

68. $f(x) = x\sinh x$

69. $f(x) = x\arctan(x)$

70. $f(x) = x\arcsin(x)$

Partial Fractions. Integrate f by partial fractions. Check answers with computer or calculator assist.

71. $f(x) = \dfrac{x+4}{x+5}$

72. $f(x) = \dfrac{x-2}{x-4}$

73. $f(x) = \dfrac{x^2+4}{(x+1)(x+2)}$

74. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)}$

75. $f(x) = \dfrac{x+4}{(x+1)(x+2)}$

76. $f(x) = \dfrac{x-1}{(x+1)(x+2))}$

77. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x+5)}$

78. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x+3)}$

79. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x-1)}$

80. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x-1)}$

Special Methods. Integrate f by using the suggested u-substitution or method. Check answers with computer or calculator assist.

81. $f(x) = \dfrac{x^2+2}{(x+1)^2}$, $u = x+1$.

82. $f(x) = \dfrac{x^2+2}{(x-1)^2}$, $u = x-1$.

83. $f(x) = \dfrac{2x}{(x^2+1)^3}$, $u = x^2+1$.

84. $f(x) = \dfrac{3x^2}{(x^3+1)^2}$, $u = x^3+1$.

85. $f(x) = \dfrac{x^3+1}{x^2+1}$, use long division.

86. $f(x) = \dfrac{x^4+2}{x^2+1}$, use long division.