

# Differential Equations and Linear Algebra

2250-4 [12:25 class] at 10:30am on 17 Dec 2009

**Instructions.** The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c),  $\dots$ . Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly eight** separately stapled packages of problems, one package per chapter. These packages will be submitted as four grading stacks, no extra staples, for grading by

Fusi [ch3,ch7], Richins [ch4,ch5], Jacobsen [ch6], Johnson [ch8], Matheson [ch9],  
Gustafson [ch10].

Each grader will add one staple to bind the chapter packages. The graded exams will be in a box outside 113 JWB; you will pick up several stapled packages.

- Electronic records will be posted at the web site for the course. Recording errors should be reported by email only.

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**Final Grade.** The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

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**Please discard this page or keep it for your records.**

## Differential Equations and Linear Algebra 2250-4 [12:25 class]

Final Exam at 10:30am on 17 Dec 2009

**Ch3. (Linear Systems and Matrices)** Complete all problems.

[10%] **Ch3(a):** Check the correct box. Incorrect answers lose all credit.

**Part 1.** [5%]:  True or  False:

If  $A$  and  $B$  are two  $2 \times 2$  triangular matrices, then  $AB$  is triangular.

**Part 2.** [5%]:  True or  False:

If matrix  $A$  has 3 rows and 2 columns, and a row of zeros, then  $A\mathbf{x} = \mathbf{0}$  has infinitely many solutions  $\mathbf{x}$ .

**Answer:** False. False.

[30%] **Ch3(b):** Determine which values of  $k$  correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system  $A\mathbf{x} = \mathbf{b}$  given by

$$A = \begin{pmatrix} 1 & 5 & -k \\ 1 & 5 & -3 \\ 0 & k-2 & k-3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ k-2 \\ -2k-5 \end{pmatrix}.$$

**Answer:** Elimination methods with swap, combo, multiply give  $\begin{pmatrix} 1 & 5 & -3 & k-2 \\ 0 & k-2 & 0 & -3k-2 \\ 0 & 0 & k-3 & k-3 \end{pmatrix}$ .

Then (1) Unique solution for  $x \neq 2$  and  $x \neq 3$ ; (2) No solution for  $k = 2$  [signal equation  $0 = -8$ ]; (3) Infinitely many solutions for  $k = 3$ .

[20%] **Ch3(c):** Let matrix  $C$  and vector  $\mathbf{b}$  be defined by the equations

$$C = \begin{pmatrix} -2 & 3 & -1 \\ 0 & -2 & 4 \\ 1 & 3 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Let  $I$  denote the  $3 \times 3$  identity matrix. Find the value of  $x_2$  by Cramer's Rule in the system  $(2I+C)\mathbf{x} = \mathbf{b}$ .

**Answer:**  $x_2 = \Delta_2/\Delta$ ,  $\Delta_2 = 6$ ,  $\Delta = \det(2I+C) = 12$ ,  $x_2 = 1/2$ .

[30%] **Ch3(d):** Display the adjugate matrix [or adjoint matrix] of  $A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$ .

**Answer:**  $\text{adj}(A) = \begin{pmatrix} -12 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

[10%] **Ch3(e):** Assume the adjugate matrix of  $A$  is  $\begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$ . Find  $A$ .

**Answer:**  $A = \begin{pmatrix} 4 & 6 \\ 0 & 2 \end{pmatrix}$

**Staple this page to the top of all Ch3 work. Submit one stapled package per chapter.**

## Differential Equations and Linear Algebra 2250-4 [12:25 class]

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**Ch4. (Vector Spaces)** Complete all problems.

[30%] **Ch4(a):** Define  $S$  to be the set of all vectors  $\mathbf{x}$  in  $\mathcal{R}^5$  such that  $x_1 + x_3 + 2x_5 = 0$  and  $x_3 + x_4 = x_1$ . Prove that  $S$  is a subspace of  $\mathcal{R}^5$ .

**Answer:** Let  $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ . Then the restriction equations can be written as

$A\mathbf{x} = \mathbf{0}$ . Apply the kernel theorem. This is theorem 2 in section 4.2 of Edwards-Penney.

[10%] **Ch4(b):** Give an example of three vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  for which the rank of their augmented matrix is one.

**Answer:** Let  $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3$  for any nonzero vector  $\mathbf{v}_1$ .

[30%] **Ch4(c):** Apply an independence test to the vectors below. Report **independent** or **dependent**.

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

**Answer:** Dependent. The rank of the augmented matrix of the three vectors is 2.

[30%] **Ch4(d):** The  $5 \times 5$  matrix  $A$  below has some independent columns. Report the independent columns of  $A$ , according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -2 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 6 & 0 & 6 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \end{pmatrix}$$

**Answer:** Find  $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ . The pivot columns are 1 and 3.

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## Differential Equations and Linear Algebra 2250-4 [12:25 class]

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**Ch5. (Linear Equations of Higher Order)** Complete all problems.

[10%] **Ch5(a):** Report the general solution  $y(x)$  of the differential equation

$$6\frac{d^2y}{dx^2} - 31\frac{dy}{dx} + 5y = 0.$$

**Answer:** Characteristic equation  $(6r - 1)(r - 5) = 0$  has roots  $1/6, 5$ . The general solution  $y(x)$  is a linear combination of the atoms  $e^{x/6}, e^{5x}$ .

[20%] **Ch5(b):** Given a damped spring-mass system  $mx''(t) + cx'(t) + kx(t) = 0$  with  $m = 25, k = 4$  and  $c > 0$  a symbol, calculate all values of symbol  $c$  such that the solution  $x(t)$  is **critically-damped**. Please, **do not solve** the differential equation!

**Answer:** The discriminant must be zero. Then  $c^2 - (4)(25)(4) = 0$ , or  $c = 20$ .

[20%] **Ch5(c):** Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that  $y = x^2 + 5e^{-x} + x \cos x$  is a solution.

**Answer:** The atoms  $x^2, e^{-x}, \cos x$  correspond to roots  $0, 0, 0, -1, i, -i$ . The factor theorem implies the characteristic polynomial should be  $r^3(r + 1)(r^2 + 1)$ .

[20%] **Ch5(d):** Determine a basis of solutions of a homogeneous constant-coefficient differential equation, given it has characteristic equation

$$r^2(r^2 + r)(r^2 + r + 1) = 0.$$

**Answer:** The roots are  $0, 0, 0, -1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$ . By Euler's theorem, a basis is the set of atoms for these roots:  $1, x, x^2, e^{-x}, e^{-x/2} \cos(\sqrt{3}x/2), e^{-x/2} \sin(\sqrt{3}x/2)$ .

[30%] **Ch5(e):** Determine the **shortest** trial solution for  $y_p$  according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 2x^2 + 3 \sin 2x + 4e^x$$

**Answer:** Let  $f(x) = 2x^2 + 3 \sin 2x + 4e^x$ . The atoms in  $f$  are  $x^2, \sin 2x, e^x$ . The completed set of atoms, including all lower-power related atoms, is  $1, x, x^2, \cos 2x, \sin 2x, e^x$ . There are 6 atoms in this list. A theorem says that the shortest trial solution contains 6 atoms. Break the 6 atoms into four groups, each with the same base atom: group 1 ==  $1, x, x^2$ ; group 2 ==  $\cos 2x$ ; group 3 ==  $\sin 2x$ ; group 4 ==  $e^x$ . Modify each group, to remove any solution of the homogeneous equation  $y^{(4)} + 4y^{(2)} = 0$ . Then the four groups are replaced by group 1\* ==  $x^2, x^3, x^4$ ; group 2\* ==  $x \cos 2x$ ; group 3\* ==  $x \sin 2x$ ; group 4\* ==  $e^x$ . The shortest trial solution is a linear combination of these last six atoms.

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**Differential Equations and Linear Algebra 2250-4 [12:25 class]**

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**Ch6. (Eigenvalues and Eigenvectors)** Complete all problems.

[30%] **Ch6(a):** Find the eigenvalues of the matrix  $A = \begin{pmatrix} 0 & -3 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 5 \end{pmatrix}$ .

To save time, **do not** find eigenvectors!

**Answer:** The characteristic polynomial is  $\det(A - rI) = (r^2 + 9)(5 - r)(r - 2)^2$ . The eigenvalues are  $2, 2, 5, \pm 3i$ . Determinant expansion is by the cofactor method along column 1. All  $3 \times 3$  minors are also expanded by the cofactor method.

[30%] **Ch6(b):** Find the eigenvectors corresponding to complex eigenvalues  $1 \pm 2i$  for the  $2 \times 2$  matrix  $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ .

**Answer:**  $\left(1 + 2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(1 - 2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

[20%] **Ch6(c):** Find a  $2 \times 2$  matrix  $A$  with eigenpairs

$$\left(4, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(-5, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

**Answer:**  $A = \begin{pmatrix} 22 & -18 \\ 27 & -23 \end{pmatrix}$

[20%] **Ch6(d):** Let  $I$  denote the  $3 \times 3$  identity matrix. Assume given two  $3 \times 3$  matrices  $B, C$ , which satisfy  $CP = PB$  for some invertible matrix  $P$ . Let  $C$  have eigenvalues  $-1, 1, 5$ . Find the eigenvalues of  $A = 2I + 3B$ .

**Answer:** Both  $B$  and  $C$  have the same eigenvalues, because  $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$ . Further, both  $B$  and  $C$  are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix  $B = \mathbf{diag}(-1, 1, 5)$ . In this case,  $A = 2I + 3B = \mathbf{diag}(2, 2, 2) + \mathbf{diag}(-3, 3, 15) = \mathbf{diag}(-1, 5, 17)$  and the eigenvalues of  $A$  are  $-1, 5, 17$ .

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## Differential Equations and Linear Algebra 2250-4 [12:25 class]

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**Ch7. (Linear Systems of Differential Equations)** Complete all problems.

[30%] **Ch7(a):** Solve for the general solution  $x(t)$ ,  $y(t)$  in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned}\frac{dx}{dt} &= x + 3y, \\ \frac{dy}{dt} &= 18x + 4y.\end{aligned}$$

**Answer:** Define  $A = \begin{pmatrix} 1 & 3 \\ 18 & 4 \end{pmatrix}$ . The eigenvalues  $-5, 10$  are roots of the characteristic equation  $\det(A - rI) = (r + 5)(r - 10) = 0$ . By Cayley-Hamilton-Ziebur,  $x(t) = c_1e^{-5t} + c_2e^{10t}$ . Using the first differential equation  $x' = x + 3y$  implies  $y(t) = \frac{1}{3}(x' - x) = -2c_1e^{-5t} + 3c_2e^{10t}$ .

[50%] **Ch7(b):** Define

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenvalues of  $A$  are  $-1, -1, 1$ . Apply the eigenanalysis method to solve the differential system  $\mathbf{u}' = A\mathbf{u}$ .

The term *eigenanalysis* refers to the process of finding eigenvalues and eigenvectors of the matrix  $A$ . After finding the eigenpairs, report the general solution  $\mathbf{u}(t)$ .

**Answer:** Form  $A_1 = A - (-1)I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ . Then  $\text{rref}(A_1) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . The last frame algorithm implies general solution of  $A_1\mathbf{x} = \mathbf{0}$  is  $x_1 = -t_1 - t_2$ ,  $x_2 = t_1$ ,  $x_3 = t_2$ . Take partials on symbols  $t_1, t_2$  to obtain the eigenvectors  $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ . Repeat with

$A_2 = A - (1)I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$  to get  $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ . The general solution of  $\mathbf{u}' = A\mathbf{u}$  is  $\mathbf{u}(t) = c_1e^{-t}\mathbf{v}_1 + c_2e^{-t}\mathbf{v}_2 + e^t\mathbf{v}_3$ .

[20%] **Ch7(c):** Find the  $2 \times 2$  matrix  $A$ , given the general solution of  $\mathbf{u}' = A\mathbf{u}$  is

$$\mathbf{u}(t) = c_1e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

**Answer:** Let  $P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ ,  $D = \text{diag}(3, 0)$ . Then  $A = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ .

**Staple this page to the top of all Ch7 work. Submit one stapled package per chapter.**

**Differential Equations and Linear Algebra 2250-4 [12:25 class]**

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**Ch8. (Matrix Exponential)** Complete all problems.[20%] **Ch8(a):** Using any method in the lectures or the textbook, display the matrix exponential

$$e^{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}t}.$$

**Answer:**  $e^{At} = \mathbf{diag}(e^{3t}, e^{-t})$  by Putzer's formula  $e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$ . The eigenvalues  $\lambda_1 = 3$ ,  $\lambda_2 = -1$  are found from  $\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) = 0$ .

[10%] **Ch8(b):** Consider the  $2 \times 2$  system

$$\begin{aligned}x' &= 3x, \\y' &= -y, \\x(0) &= 1, \quad y(0) = 2.\end{aligned}$$

Solve the system  $\mathbf{u}' = A\mathbf{u}$  for  $\mathbf{u}$ , using  $e^{At}$  from the previous problem.

**Answer:**  $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{-t} \end{pmatrix}.$

[40%] **Ch8(b):** Display the matrix form of variation of parameters for the  $2 \times 2$  system. Then integrate to find one particular solution.

$$\begin{aligned}x' &= 3x + 2, \\y' &= -y + 1.\end{aligned}$$

**Answer:**  $\mathbf{u}_p(t) = e^{At} \int_0^t e^{-As} \begin{pmatrix} 2 \\ 1 \end{pmatrix} ds$ . Then  $e^{At} = \mathbf{diag}(e^{3t}, e^{-t})$  from the first problem,

$$e^{-As} = \mathbf{diag}(e^{-3s}, e^s), \text{ and } \int_0^t \mathbf{diag}(e^{-3s}, e^s) \begin{pmatrix} 2 \\ 1 \end{pmatrix} ds = \int_0^t \mathbf{diag}(2e^{-3s}, e^s) ds = \begin{pmatrix} \frac{2}{3} - \frac{2}{3}e^{-3t} \\ e^t - 1 \end{pmatrix}.$$

Finally,  $\mathbf{u}_p(t) = \begin{pmatrix} -\frac{2}{3} + \frac{2}{3}e^{3t} \\ 1 - e^{-t} \end{pmatrix}.$

[30%] **Ch8(c):** Check the correct statements.

1. The system  $\mathbf{u}' = A\mathbf{u}$  can be solved even when  $A$  is not diagonalizable.
2. The general solution of  $\mathbf{u}' = A\mathbf{u}$  can be written as  $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$ .
3. For any  $n \times n$  matrix  $A$ ,  $(sI - A)^{-1}$  equals the Laplace integral of  $e^{At}$ .

**Answer:** All three are true.

**Staple this page to the top of all Ch8 work. Submit one stapled package per chapter.**

## Differential Equations and Linear Algebra 2250-4 [12:25 class]

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**Ch9. (Nonlinear Systems)** Complete all problems.

[30%] **Ch9(a):**

Determine whether the equilibrium  $\mathbf{u} = \mathbf{0}$  is stable or unstable. Then classify the equilibrium point  $\mathbf{u} = \mathbf{0}$  as a saddle, center, spiral or node.

$$\mathbf{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \mathbf{u}$$

**Answer:** The eigenvalues of  $A$  are roots of  $r^2 - 2r + 5 = (r - 1)^2 + 4 = 0$ , which are complex conjugate roots  $1 \pm 2i$ . Rotation eliminates the saddle and node. Finally, the atoms  $e^t \cos 2t$ ,  $e^t \sin 2t$  have limit zero at  $t = -\infty$ , therefore the system is unstable at  $t = \infty$ . So it must be a spiral [centers are stable]. Report: **unstable spiral**.

[30%] **Ch9(b):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y, \\ y' &= x - y + 11 - x^2. \end{aligned}$$

An equilibrium point is  $x = \sqrt{11}$ ,  $y = \sqrt{11}$ . Compute the Jacobian matrix  $A$  of the linearized system at this equilibrium point.

**Answer:** The Jacobian is  $J(x, y) = \begin{pmatrix} 1 & -1 \\ 1 - 2x & -1 \end{pmatrix}$ . Then  $A = J(\sqrt{11}, \sqrt{11}) = \begin{pmatrix} 1 & -1 \\ 1 - 2\sqrt{11} & -1 \end{pmatrix}$ .

[40%] **Ch9(c):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x + y, \\ y' &= 2x + 2y + 8 - 2x^2. \end{aligned}$$

At equilibrium point  $x = -2$ ,  $y = 2$ , the Jacobian matrix is  $A = \begin{pmatrix} 1 & 1 \\ 10 & 2 \end{pmatrix}$ .

- (1) Determine the stability at  $t = \infty$  and the phase portrait classification at  $\mathbf{u} = \mathbf{0}$  for the linear system  $\mathbf{u}' = A\mathbf{u}$ .
- (2) Apply a theorem to classify  $x = -2$ ,  $y = 2$  as a saddle, center, spiral or node for **the nonlinear dynamical system**.

**Answer:** The Jacobian is  $J(x, y) = \begin{pmatrix} 1 & 1 \\ 2 - 4x & 2 \end{pmatrix}$ . Then  $A = J(-2, 2) = \begin{pmatrix} 1 & 1 \\ 2 - 4(-2) & 2 \end{pmatrix}$ .

The eigenvalues of  $A$  are found from  $r^2 - 3r - 8 = 0$ , giving roots  $a = 4.7$ ,  $b = -1.7$ , approximately. Because both atoms  $e^{at}$ ,  $e^{bt}$  are real, rotation does not happen, and the classification must be a saddle or a node. The atoms don't limit to zero at either  $t = -\infty$  or  $t = \infty$ , therefore it can't be a node, and we report an **unstable saddle** for the linear problem  $\mathbf{u}' = A\mathbf{u}$  at equilibrium  $\mathbf{u} = \mathbf{0}$ . Theorem 2 in 9.2 applies to say that the same is true for the nonlinear system: **unstable saddle** at  $x = -2$ ,  $y = 2$ .

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**Differential Equations and Linear Algebra 2250-4 [12:25 class]**

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**Ch10. (Laplace Transform Methods)** Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[10%] **Ch10(a):** Fill in the blank spaces in the minimum backward Laplace table:

|                     |               |                 |                 |                 |                     |                     |
|---------------------|---------------|-----------------|-----------------|-----------------|---------------------|---------------------|
| $f(t)$              |               |                 |                 |                 |                     |                     |
| $\mathcal{L}(f(t))$ | $\frac{1}{s}$ | $\frac{1}{s^2}$ | $\frac{1}{s^n}$ | $\frac{1}{s-a}$ | $\frac{s}{s^2+b^2}$ | $\frac{b}{s^2+b^2}$ |

[20%] **Ch10(b):** Compute  $\mathcal{L}(f(t))$  for  $f(t) = t$  on  $2 \leq t < 3$ ,  $f(t) = 0$  otherwise.

**Answer:** Use  $f(t) = tg(t)$ ,  $g(t) = \mathbf{pulse}(t, 2, 3) = \mathbf{step}(t-2) - \mathbf{step}(t-3)$ , and the second shifting theorem. Then  $\mathcal{L}(f(t)) = \mathcal{L}(t\mathbf{step}(t-2)) - \mathcal{L}(t\mathbf{step}(t-3)) = e^{-2s}\mathcal{L}(t+2) - e^{-3s}\mathcal{L}(t+3) = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) - e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

[20%] **Ch10(c):** Solve for  $f(t)$  in the equation  $\mathcal{L}(f(t)) = \frac{e^{-s}}{s+4}$ .

**Answer:** Use the second shifting theorem,  $e^{-as}\mathcal{L}(h(t)) = \mathcal{L}(h(t-a)\mathbf{step}(t-a))$ . Then  $\mathcal{L}(f(t)) = e^{-s}\mathcal{L}(e^{-4t}) = \mathcal{L}(e^{-4u}|_{u=t-1}\mathbf{step}(t-1)) = \mathcal{L}(e^{4-4t}\mathbf{step}(t-1))$ . Lerch's theorem implies  $f(t) = e^{4-4t}\mathbf{step}(t-1)$ .

[20%] **Ch10(d):** Solve for  $f(t)$  in the equation  $\mathcal{L}(f(t)) = \frac{1}{(s+1)^3} + \frac{d}{ds}\mathcal{L}(\cos t)$ .

**Answer:**  $\mathcal{L}(f) = \frac{1}{u^3}|_{u=s+1} + \mathcal{L}((-t)\cos t) = \mathcal{L}(e^{-t}\left(\frac{t^2}{2}\right) - t\cos t)$ , giving the final answer  $f(t) = e^{-t}\left(\frac{t^2}{2}\right) - t\cos t$ . The Laplace details use the first shifting theorem and  $s$ -differentiation theorem.

[30%] **Ch10(e):** Solve by Laplace's method for the solution  $x(t)$ :

$$x''(t) + 2x'(t) = 8e^{2t}, \quad x(0) = x'(0) = 0.$$

**Answer:**  $x(t) = e^{2t} + e^{-2t} - 2$ . The Laplace steps are: (1)  $\mathcal{L}(x(t)) = \frac{8}{s(s+2)(s-2)}$ ; (2)  $\mathcal{L}(x(t)) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-2} = \mathcal{L}(A + Be^{-2t} + Ce^{2t})$  where by Heaviside's shortcut the partial fraction answers are  $A = -2$ ,  $B = 1$ ,  $C = 1$ .

**Staple this page to the top of all Ch10 work. Submit one stapled package per chapter.**