Differential Equations and Linear Algebra

2250-1 [7:30 class] at 7:30am on 15 Dec 2009

KEY

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$. The credits should add to 100. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. There may be replacement problems at reduced credit, in case the problem (a), (b), (c), $\cdots$ cannot be solved. The number of graded problems is fixed. For instance, if ch3(a) and ch3(a_1) are solved, then ch3(a) is ignored and its replacement ch3(a_1) will be graded instead. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly eight** separately stapled packages of problems, one package per chapter. These packages will be submitted as four grading stacks, no extra staples, for grading by

  Fusi [ch3,ch7], **Richins** [ch4,ch5], **Jacobsen** [ch6], **Johnson** [ch8], **Matheson** [ch9], **Gustafson** [ch10].

  Each grader will add one staple to bind the chapter packages. The graded exams will be in a box outside 113 JWB; you will pick up several stapled packages.
- Electronic records will be posted at the web site for the course. Recording errors should be reported by email only.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90+91+92+89+89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please discard this page or keep it for your records.

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Ch3. (Linear Systems and Matrices) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [40%] **Ch3(a)**: This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$, a topic in Section 3.6 of Edwards-Penney.

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $C^{-1}B$.

$$B = \begin{pmatrix} 1 & -1 & ? & ? \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 2 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

☐ [40%] **Ch3(b)**: **No replacement. This problem is required.** Determine which values of k correspond to (1) a unique solution, (2) infinitely many solutions and (3) no solution, for the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & 3 & -k \\ 0 & k-2 & k-4 \\ 1 & 3 & -4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2k-6 \\ k-3 \end{pmatrix}.$$

☐ [20%] **Ch3(c)**: Let matrix C and vector $\mathbf{b}$ be defined by the equations

$$C = \begin{pmatrix} -1 & 3 & -1 \\ 0 & -1 & 4 \\ 1 & 3 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Let I denote the 3×3 identity matrix. Find the value of x_2 by Cramer's Rule in the system $(I+C)\mathbf{x} = \mathbf{b}$.

If you finished Ch3(a), Ch3(b) and Ch3(c), then 100% has been marked – go on to Ch4. Otherwise, you may solve **Ch3**(a_1) and/or **Ch3**($\cancel{b_1}$). Graders will mark **Ch3(a)** as zero, if you post a solution to **Ch3**(a_1). The same for **Ch3(c)** and **Ch3**(c_1). There is reduced credit for each replacement.

☐ [30%] **Ch3**(a_1): Find the adjugate of A and the inverse of $A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$.

20% c_1

☐ [30%] **Ch3**(b_1):

Part I [10%]: State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$.

Part II [10%]: Give an example of two 2×2 matrices A and B such that AB is **lower** triangular.

Part III [10%]: Give an example of a matrix A with 3 rows and 2 columns such that $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x}$.

Staple this page to the top of all Ch3 work. Submit one stapled package per chapter.

[40%] Ch3(a) $BC = \begin{pmatrix} |B| & 0 & 0 & 0 \\ 0 & |B| & 0 & 0 \\ 0 & 0 & |B| & 0 \\ 0 & 0 & 0 & |B| \end{pmatrix} \Rightarrow$ row(B,1) Col(C,1) $= |B| \Rightarrow 8 = |B|$.

$|C^{-1}B| = \dfrac{|B|}{|C|}$ and $|BC| = |B|^4 \Rightarrow |C^{-1}B| = \dfrac{|B|}{|B|^3} = \boxed{\dfrac{1}{64}}$

[40%] Ch3(b) $\left(\begin{array}{ccc|c} 1 & 3 & -k & 1 \\ 0 & k-2 & k-4 & -2k-6 \\ 1 & 3 & -4 & k-3 \end{array}\right)$

$\left(\begin{array}{ccc|c} 1 & 3 & -k & 1 \\ 0 & k-2 & k-4 & -2k-6 \\ 0 & 0 & k-4 & k-4 \end{array}\right)$ combo(1,3,−1)

$\left(\begin{array}{ccc|c} 1 & 3 & -k & 1 \\ 0 & k-2 & 0 & -3k-2 \\ 0 & 0 & k-4 & k-4 \end{array}\right)$ combo(3,2,−1)

unique sol: $(k-2)(k-4) \neq 0$ (Three lead variables)
∞-many sols: $k-4=0$ (free variable z)
No sol: $k-2=0$ (signal of $0=-8$)

[20%] Ch3(c) $x_2 = \dfrac{\Delta_2}{\Delta} = \boxed{\dfrac{12}{20}}$

$\Delta = \begin{vmatrix} -1 & 3 & -1 \\ 0 & -1 & 4 \\ 1 & 3 & -3 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 4 \\ 3 & -3 \end{vmatrix} + (1)\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix}$

$\Delta = 9 + 11 = 20$

$\Delta_2 = \begin{vmatrix} -1 & 1 & -1 \\ 0 & 0 & 4 \\ 1 & 2 & -3 \end{vmatrix} = (-1)(4)\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = 12$

[30%] ch3(a₁) $\operatorname{adj}(A) = \begin{pmatrix} -12 & 4 & 0 \\ 1 & 1 & 2 \\ 8 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} -12 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

$|A| = 8$ $\quad A^{-1} = \frac{1}{8}\operatorname{adj}(A)$

[20%] ch3(c₁) part I. (1) unique sol. (2) ∞-many sols (3) No sol.

part II, $A = B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ product of lower triangular is also lower triangular

part III. $\begin{cases} x = 0 \\ y = 0 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array} \Leftrightarrow A\vec{u} = \vec{0}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

unique sol $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

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Ch4. (Vector Spaces) Complete all problems. **No replacements. All problems required.**

[30%] **Ch4(a):** Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1+x_3+2x_4=0$ and $x_3+x_4=x_2$. Prove that S is a subspace of $\mathcal{R}^4$.

[10%] **Ch4(b):** Independence of 4 fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ can be decided by computing the rank of their augmented matrix. State a **different** test which can decide upon independence of four vectors in $\mathcal{R}^4$.

[30%] **Ch4(c):** Apply an independence test to the vectors below. Report **independent** or **dependent**.

$$\mathbf{v}_1=\begin{pmatrix}-1\\1\\2\\0\end{pmatrix},\quad \mathbf{v}_2=\begin{pmatrix}3\\0\\1\\0\end{pmatrix},\quad \mathbf{v}_3=\begin{pmatrix}4\\-1\\-1\\0\end{pmatrix}.$$

[30%] **Ch4(d):** The 5×5 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A=\begin{pmatrix}0&0&0&0&0\\0&-3&-2&0&1\\0&-1&0&0&1\\0&6&6&0&0\\0&2&2&0&0\end{pmatrix}$$

Staple this page to the top of all Ch4 work. Submit one stapled package per chapter.

[30%] Ch 4(a) Define $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The restriction equations are equivalent to $A\vec{x} = \vec{0}$. Apply the kernel Theorem, Thm 2 in Section 4.2.

[10%] Ch 4(b) Test: det of the augmented matrix is not zero $\Longleftrightarrow$ independence

[30%] Ch 4(c) $\begin{pmatrix} -1 & 3 & 4 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 4 \\ 0 & 3 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ rank $= 2 \Rightarrow$ dependent.

[30%] Ch 4(d) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix} \xrightarrow[\text{swap}]{} \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{swaps}]{} \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & -3 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ combo

$\to \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Cols 2, 3 are independent by The pivot Thm

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Ch5. (Linear Equations of Higher Order) Complete all problems. **No replacements. All problems required.**

[10%] **Ch5(a):** Report the general solution $y(x)$ of the differential equation

$$6\frac{d^2y}{dx^2} + 31\frac{dy}{dx} + 5y = 0.$$

[20%] **Ch5(b):** Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 2$, $k = 5$ and $c > 0$ a symbol, calculate all values of symbol c such that the solution $x(t)$ is **under-damped**. Please, **do not solve** the differential equation!

[20%] **Ch5(c):** Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = x^2 + 5e^{-x} + x\cos x$ is a solution.

[20%] **Ch5(d):** Determine a basis of solutions of a homogeneous constant-coefficient differential equation, given it has characteristic equation

$$r^2(r^2 + r)(r^2 + r + 1) = 0.$$

[30%] **Ch5(e):** Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 2x^2 + 3\sin 2x + 4e^x$$

Staple this page to the top of all Ch5 work. Submit one stapled package per chapter.

[10%] Ch5(a) $6r^2+31r+5 = (6r+1)(r+5)$

$y = c_1 e^{-x/6} + c_2 e^{-5x}$

[20%] Ch5(b) $c^2 - 4mk < 0 \Rightarrow c^2 < 4km \Rightarrow c < \sqrt{4(2)(5)} = \sqrt{40}$

$\boxed{0 < c < 2\sqrt{10}}$

[20%] Ch5(c) atoms $= 1, x, x^2, e^{-x}, \cos x, x\cos x, \sin x, x\sin x$

roots $= 0, 0, 0, -1, \pm i, \pm i$

Char. eq. $= r^3(r+1)(r^2+1)^2$

[20%] Ch5(d) atoms = Basis $= 1, x, x^2, e^{-x}, e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right), e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)$

$r^3(r+1)(r^2+r+1)=0$

$r = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\cdot1\cdot1} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$

[30%] Ch5(e) $f(x) = 2x^2 + 3\sin 2x + 4e^x$

atoms of $f(x) = x^2, \sin 2x, e^x$

completed list $= 1, x, x^2, \cos 2x, \sin 2x, e^x$

Shortest trial sol y has 6 atoms

group 1 $\boxed{1, x,}\ x^2, x^3, x^4$

removed

group 2 $\boxed{\cos 2x},\ x\cos 2x$

removed

group 3 $\boxed{\sin 2x},\ x\sin 2x$

removed

group 4 e^x

y = linear combination of $x^2, x^3, x^4,$

$x\cos 2x,$

$x\sin 2x,$

e^x

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems. **All problems required. No replacement.**

[30%] **Ch6(a):** Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 4 & -5 & 0 & 0 \\ -4 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

[30%] **Ch6(b):** Find the eigenvectors corresponding to complex eigenvalues $1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

[20%] **Ch6(c):** Find a 2×2 matrix A with eigenpairs

$$\left(4, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \left(-5, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

[20%] **Ch6(d):** Assume two 3×3 matrices B, C are related by $CP = PB$ where P is invertible. Let C have eigenvalues -1, 1, 6. Find the eigenvalues of $A = (1/4)B + 2I$, where I is the identity matrix.

Staple this page to the top of all Ch6 work. Submit one stapled package per chapter.

[30%] Ch 6 (a) $\det(A-\lambda I) = (\lambda^2+16)(3-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix}$

$= (\lambda^2+16)(3-\lambda)(\lambda^2-4\lambda+4)$

$= (\lambda^2+16)(3-\lambda)(\lambda-2)^2$

$\boxed{\lambda = 2, 2, 3, 4i, -4i}$

[30%] Ch6 (b) $A-(1+2i)I = \begin{pmatrix} 1-1-2i & 2 \\ -2 & 1-1-2i \end{pmatrix} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}$

$\rightarrow \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \xrightarrow{\text{mult}} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \xrightarrow{\text{swap}} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ combo(1, 2, $-i$)

$\begin{cases} x = -i\,t_1 \\ y = t_1 \end{cases}$ last frame algorithm $\Rightarrow$ $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ [partial on t_1]

Thm $\Rightarrow$ $\vec{v}_2 =$ conjugate of $\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

$\boxed{\left(1+2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(1-2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)}$

[20%] Ch6(c) $A = PDP^{-1}$ $P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 \\ 0 & -5 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\frac{1}{1}$

$A = \begin{pmatrix} 22 & -18 \\ 27 & -23 \end{pmatrix}$

[20%] Ch6 (d) $|A-\lambda I| = |\frac{1}{4}B + 2I - \lambda I| = |\frac{1}{4}I|\,|B + 8I - 4\lambda I|$

$= (\frac{1}{4})^3\,|B - (4\lambda - 8)I|$

Therefore, $4\lambda - 8 =$ eigenvalue of $B =$ eigenvalue of $C = -1, 1, 6$

$\lambda = \frac{1}{4}(8-1), \frac{1}{4}(8+1), \frac{1}{4}(8+16) = \boxed{\frac{7}{4}, \frac{9}{4}, \frac{14}{4}}$

Thm $|B-\mu I| = |P(B-\mu I)P^{-1}| = |PBP^{-1} - \mu I| = |C - \mu I|$

$\Rightarrow$ B and C have the same eigenvalues.

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Ch7. (Linear Systems of Differential Equations) Complete all problems. **No replacement. All problems required.**

[30%] **Ch7(a):** Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\begin{aligned}\frac{dx}{dt} &= x + 3y,\\ \frac{dy}{dt} &= 18x + 4y.\end{aligned}$$

[50%] **Ch7(b):** Apply the eigenanalysis method to solve the differential system $\mathbf{u}' = A\mathbf{u}$, given

$$A = \begin{pmatrix} -3 & 5 & -10 \\ 0 & 2 & 0 \\ 5 & -5 & 12 \end{pmatrix}$$

The eigenvalues of A are 2, 2, 7. The term *eigenanalysis* refers to the process of finding eigenvalues and eigenvectors of the matrix A. After finding the eigenpairs, report the general solution $\mathbf{u}(t)$.

[20%] **Ch7(c):** Assume A is a 2×2 matrix and the general solution of $\mathbf{u}' = A\mathbf{u}$ is given by

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Find the 2×2 matrix A.

Staple this page to the top of all Ch7 work. Submit one stapled package per chapter.

[30%] Ch 7(a) Cayley-Hamilton-Ziebur $\Rightarrow$ $x(t)$ = linear combination of atoms determined from roots of $\lambda^2 - 5\lambda - 50 = 0$

$(\lambda+5)(\lambda-10) = 0 \Rightarrow \lambda = -5, 10$

$$\boxed{x(t) = c_1 e^{-5t} + c_2 e^{10t}}$$

$$\boxed{y(t) = -2c_1 e^{-5t} + 3c_2 e^{10t}}$$

$y = \frac{x' - x}{3}$ from the first DE

$= \frac{-5c_1 e^{-5t} + 10c_2 e^{10t} - c_1 e^{-5t} - c_2 e^{10t}}{3}$

$= \frac{-6}{3} c_1 e^{-5t} + \frac{9}{3} c_2 e^{10t}$

[50%] Ch 7(b)

$$A - 2I = \begin{pmatrix} -5 & 5 & -10 \\ 0 & 0 & 0 \\ 5 & -5 & 10 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}$$

$\begin{cases} x = t_1 - 2t_2 \\ y = t_1 \\ z = t_2 \end{cases}$ last frame algorithm $\Rightarrow$ $\vec{v}_1 = \partial_{t_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \partial_{t_2} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$$A - 7I = \begin{pmatrix} -10 & 5 & -10 \\ 0 & -5 & 0 \\ 5 & -5 & 5 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

$\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x = -t_1 \\ y = 0 \\ z = t_1 \end{cases}$ last frame alg. $\Rightarrow$ $\vec{v}_3 = \partial_{t_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Eigenpairs = $\left(2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \left(2, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right), \left(7, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)$

$$\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

[20%] Ch 7(c) Eigenpairs = $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(0, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \frac{1}{3}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \frac{1}{3} = \boxed{\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}}$$

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Ch8. (Matrix Exponential) Complete all problems. **No replacement. All problems required.**

[30%] **Ch8(a):** Using any method in the lectures or the textbook, display the matrix exponential e^{At} for the 2×2 system. Then go on to solve the system $\mathbf{u}' = A\mathbf{u}$ for $\mathbf{u}$.

$$\begin{aligned} x' &= 3x, \\ y' &= -y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

To save time, find e^{At} explicitly, because the answer is used in the next problem.

[40%] **Ch8(b):** Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find a particular solution.

$$\begin{aligned} x' &= 3x + 2, \\ y' &= -y + 1. \end{aligned}$$

[30%] **Ch8(c):** Check the correct statements.

- [ ] **1.** The general solution of $\mathbf{u}' = A\mathbf{u}$ is expressed in term of eigenpairs of A, even when A is not diagonalizable.
- [x] **2.** The general solution of $\mathbf{u}' = A\mathbf{u}$ can be written as $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$.
- [x] **3.** For any $n \times n$ matrix A, the Laplace integral of e^{At} equals the inverse of $sI - A$.

Staple this page to the top of all Ch8 work. Submit one stapled package per chapter.

[30%] Ch 8(a) $\quad A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \qquad \lambda_1 = 3,\ \lambda_2 = -1$

$$e^{At} = e^{3t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{3-(-1)}(A - 3I)$$

$$= e^{3t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}\right)\begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

or use

<u>Thm</u> $e^{\mathrm{diag}(a,b)\,t} = \mathrm{diag}(e^{at}, e^{bt})$

[40%] Ch 8(b) $\quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\vec{u}_p = e^{At}\int_0^t e^{-As}\begin{pmatrix} 2 \\ 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-3s} & 0 \\ 0 & e^{s} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\int_0^t \begin{pmatrix} 2e^{-3s} \\ e^{s} \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{2}{-3}(e^{-3t}-1) \\ e^{t}-1 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{2}{3} + \frac{2}{3}e^{3t} \\ 1 - e^{-t} \end{pmatrix}}$$

[30%] Ch 8(c)

1. False. The formula covers only the diagonalizable case
2. Basic Theorem, Ch 8.
3. Lecture notes on Laplace resolvent

Name KEY

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Differential Equations and Linear Algebra 2250-1 [7:30 class]

Final Exam at 7:30am on 15 Dec 2009

Ch9. (Nonlinear Systems) Complete all problems. **No replacement. All problems required.**

[30%] **Ch9(a):**
Determine whether the equilibrium $\mathbf{u} = \mathbf{0}$ is stable or unstable. Then classify the equilibrium point $\mathbf{u} = \mathbf{0}$ as a saddle, center, spiral or node.

$$\mathbf{u}' = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} \mathbf{u}$$

[30%] **Ch9(b):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y, \\ y' &= x - y + 5 - x^2. \end{aligned}$$

An equilibrium point is $x = \sqrt{5}$, $y = \sqrt{5}$. Compute the Jacobian matrix A of the linearized system at this equilibrium point.

[40%] **Ch9(c):** Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x + y, \\ y' &= 2x + 2y + x^2 - 1. \end{aligned}$$

At equilibrium point $x = -1$, $y = 1$, the Jacobian matrix is $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification for $\mathbf{u}' = A\mathbf{u}$ at $(0, 0)$.

(2) Apply a theorem to classify $x = -1$, $y = 1$ as a saddle, center, spiral or node for **the nonlinear system**.

Staple this page to the top of all Ch9 work. Submit one stapled package per chapter.

[30%] ch 9 (a) $\lambda^2+2\lambda+5=0$, $\lambda = -1 \pm 2i$, stable spiral

[30%] ch 9 (b) Jacobian $= \begin{pmatrix} 1 & -1 \\ 1-2x & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & -1 \\ 1-2\sqrt{5} & -1 \end{pmatrix}}$

[40%] ch 9 (c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \to \lambda^2 - 3\lambda + 2 = 0 \to (\lambda-2)(\lambda-1)=0$

(1) atoms $= e^{2t}, e^{t} \Rightarrow \boxed{\text{unstable Node}}$

(2) Thm 2 in 9.2 says $(-1,1)$ is an unstable node. only exception is equal real roots or purely complex roots.

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Differential Equations and Linear Algebra 2250-1 [7:30 class]

Final Exam at 7:30am on 15 Dec 2009

Ch10. (Laplace Transform Methods) Complete all problems. **No replacement. All problems required.**

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[10%] **Ch10(a)**: Fill in the blank spaces in the minimum forward Laplace table:

$f(t)$	1	t	t^n	e^{at}	$\cos bt$	$\sin bt$
$\mathcal{L}(f(t))$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{s}{s^2+b^2}$	$\frac{b}{s^2+b^2}$

[20%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for $f(t) = t$ on $1 \le t < 2$, $f(t) = 0$ otherwise.

[20%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{e^{-s}}{s^2}$.

[20%] **Ch10(d)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{(s-1)^3} + \dfrac{d}{ds}\mathcal{L}(\sin t)$.

[30%] **Ch10(e)**: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) - x'(t) = 8e^{-t}, \quad x(0) = x'(0) = 0.$$

Staple this page to the top of all Ch10 work. Submit one stapled package per chapter.

[20%] ch 10 (b) $f(t) = t \text{ pulse}(t, 1, 2) = t \text{ step}(t-1) - t \text{ step}(t-2)$

$$\mathcal{L}(f_1) = \mathcal{L}(t \text{ step}(t-1)) = e^{-s}\mathcal{L}(t+1) = e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$\mathcal{L}(f_2) = \mathcal{L}(t \text{ step}(t-2)) = e^{-2s}\mathcal{L}(t+2) = e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$$

Thm $\mathcal{L}(g(t)\text{step}(t-a)) = e^{-as}\mathcal{L}(g(t+a))$

$$\mathcal{L}(f) = \mathcal{L}(f_1) - \mathcal{L}(f_2) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$$

[20%] ch 10 (c) $\mathcal{L}(f) = e^{-s}\frac{1}{s^2} = e^{-s}\mathcal{L}(t) = \mathcal{L}((t-1)\text{ step}(t-1))$

use second shift Thm : $\mathcal{L}(g(t-a)\text{ step}(t-a)) = e^{-as}\mathcal{L}(g(t))$

$$\boxed{f(t) = (t-1)\text{ step}(t-1)}$$

[20%] ch 10 (d) $\mathcal{L}(f) = \frac{1}{(s-1)^3} + \frac{d}{ds}\mathcal{L}(\sin t)$

$$= \frac{1}{u^3}\Big|_{u=s-1} + \mathcal{L}((-t)\sin t)$$

$$= \mathcal{L}(t^2/2)\Big|_{s\to s-1} + \mathcal{L}((-t)\sin t)$$

$$= \mathcal{L}(e^t t^2/2 + (-t)\sin t)$$

$$\boxed{f(t) = \frac{1}{2}t^2e^t - t\sin t}$$

[30%] ch 10 (e) $(s^2 - s)\mathcal{L}(x) = \frac{8}{s+1}$

$$\mathcal{L}(x) = \frac{8}{(s-1)(s+1)s} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s}$$

$$= \mathcal{L}(Ae^t + Be^{-t} + C)$$

$$\boxed{\begin{array}{c} x(t) = Ae^t + Be^{-t} + C \\ A = 4 \\ B = 4 \\ C = -8 \end{array}}$$

$8 = As(s+1) + Bs(s-1) + C(s-1)(s+1)$

$s=0: \quad 8 = -C$

$s=1: \quad 8 = 2A$

$s=-1: \quad 8 = 2B$