1. (Chapter 5) Complete all.

(1a) [70%] Write the solution of $x''(t) + 9x(t) = 120 \sin(t)$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don’t convert to phase-amplitude form.**

**Answer:**

$x(t) = 15 \sin(t) - 5 \sin(3t)$

(1b) [30%] Determine the practical resonance frequency $\omega$ for the electrical equation $13I'' + 2I' + 39I = 100\omega \cos(\omega t)$.

**Answer:**

$\omega = \sqrt{I/(LC)} = \sqrt{3}$. 

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Chapter 5) Complete all.

(2a) [75%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 7 with roots 0, 0, −1, −1, −1, 2i, −2i, listed according to multiplicity. The corresponding non-homogeneous equation for unknown \( y(x) \) has right side \( f(x) = 3e^x + 4e^{-x} + 5x^3 + 6\sin 2x \). Determine the undetermined coefficients shortest trial solution for \( y_p \). To save time, do not evaluate the undetermined coefficients and do not find \( y_p(x) \)! Undocumented detail or guessing earns no credit.

Answer:

The atoms of \( f(x) \) are \( e^x, e^{-x}, x^3, \sin 2x \). Complete the list, adding the related atoms, to obtain 5 groups, each group having exactly one base atom: (1) \( e^x \), (2) \( e^{-x} \), (3) \( 1, x, x^2, x^3 \), (4) \( \cos 2x \), (5) \( \sin 2x \). The trial solution is a linear combination of 8 atoms, modified by rules to the new list (1) \( e^x \), (2) \( x^3e^{-x} \), (3) \( x^2, x^3, x^4, x^5 \), (4) \( x \cos 2x \), (5) \( x \sin 2x \).

(2b) [25%] Let \( f(x) = 4x^3e^x \). Find a constant-coefficient linear homogeneous differential equation of smallest order which has \( f(x) \) as a solution.

Answer:

The atom \( x^3e^x \) is constructed from roots 1, 1, 1, 1, listed according to multiplicity. Then the characteristic polynomial must include factor \( (r-1)^4 \). The smallest order characteristic polynomial must be a constant multiple of \( (r-1)^4 = r^4 - 4r^3 + 6r^2 - 4r + 1 \). This characteristic equation belongs to the differential equation \( y^{(4)} - 4y''' + 6y'' - 4y' + y = 0 \).

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3. (Chapter 10) Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Display the details of Laplace’s method to solve the system for \( y(t) \). Don’t waste time solving for \( x(t) \)!

\[
\begin{align*}
  x' &= x + 2y , \\
  y' &= 3x , \\
  x(0) &= 0 , \quad y(0) = 5 .
\end{align*}
\]

**Suggestion:** Solve it with scalar methods.

**Alternate method:** Laplace resolvent equation \((sI - A)L(u) = u(0)\) and Cramer’s Rule. Notation: \( u \) is the vector solution of \( u' = Au \) with components \( x(t) , y(t) \).

**Answer:**

The Laplace resolvent equation can be written out to find the relations for \( L(x(t)) , L(y(t)) \). Cramer’s rule applies to find 
\[
L(y(t)) = \frac{5(s-1)}{(s-3)(s+2)}.
\]
Then partial fractions and backward table methods determine 
\[
y(t) = 3e^{-2t} + 2e^{3t} .
\]
The same method applies to determine 
\[
L(x(t)) = \frac{10}{(s-3)(s+2)}
\] and then 
\[
x(t) = 2e^{3t} - 2e^{-2t} .
\]

(3b) [25%] Find \( f(t) \) by partial fraction methods, given

\[
L(f(t)) = \frac{3s + 15}{s(s-3)} .
\]

**Answer:**

\[
L(f(t)) = \frac{-5}{s} + \frac{8}{s-3} = L(-5 + 8e^{3t})
\]

(3c) [25%] Solve for \( f(t) \), given

\[
\frac{d^2}{ds^2}L(f(t)) = \frac{2}{s^3} + \frac{2}{(s-3)(s^2 - 6s + 9)} .
\]

**Answer:**

Use the \( s \)-differentiation theorem and the first shifting theorem to get 
\[(-t)^2 f(t) = t^2 + t^2 e^{3t} \] or \( f(t) = 1 + e^{3t} \).
4. (Chapter 10) Complete all parts.

(4a) [50%] Fill in the blank spaces in the Laplace table:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$t^3$</th>
<th>$e^{-t} \sin \pi t$</th>
<th>$2te^{-t/3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}(f(t))$</td>
<td>$\frac{6}{s^4}$</td>
<td>$\frac{1}{(2s + 5)^2}$</td>
<td>$\frac{s + 1}{s^2 + 2s + 5}$</td>
</tr>
</tbody>
</table>

Answer:
Left to right: $\frac{1}{4}te^{-5t/2}$, $e^{-t} \cos 2t$, $\frac{\pi}{(s+1)^2 + \pi^2}$, $\frac{2}{(s+1/3)^2}$.

(4b) [50%] Solve by Laplace’s method for the solution $x(t)$:

$$x''(t) + 4x(t) = 8e^{-2t}, \quad x(0) = x'(0) = 0.$$  

Answer:

$$x(t) = \sin(2t) - \cos(2t) + e^{-2t}.$$
5. (Chapter 6) Complete all parts.

(5a) [20%] Find the eigenvalues of the matrix \( A = \begin{pmatrix} -1 & 6 & 1 & 12 \\ -2 & 7 & -3 & 15 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & -5 & 2 \end{pmatrix} \). To save time, do not find eigenvectors!

Answer:
1, 5, 2 ± 5i

(5b) [40%] Given \( A = \begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \), which has eigenvalues 1, -1, -1, find all eigenvectors.

Answer:
Two frame sequences are required, one for \( \lambda = 1 \) and one for \( \lambda = -1 \). Sequence 1 starts with \( \begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \), the last frame having just one row of zeros. There is one invented symbol \( t_1 \) in the last frame algorithm answer. Taking \( \partial_{t_1} \) gives one eigenvector, \( \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \). Sequence 2 starts with \( \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \), with \( \text{rref} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). There is one invented symbol \( t_1 \) in the last frame algorithm answer. Taking \( \partial_{t_1} \) gives one eigenvector, \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). This matrix has no Fourier model, it is not diagonalizable.

(5c) [20%] Suppose a 2 x 2 matrix \( A \) has eigenpairs \( (e^2, \begin{pmatrix} 1 \\ 3 \end{pmatrix}) \), \( (e^3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \). Display an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \).

Answer:
Define \( P = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \), \( D = \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \). Then \( AP = PD \).

(5d) [20%] Assume the vector general solution \( \mathbf{u}(t) \) of the 2 x 2 linear differential system \( \mathbf{u}' = C \mathbf{u} \) is given by
\[
\mathbf{u}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} .
\]

Find the eigenpairs of the matrix \( C \).

Answer:
The missing exponential in the second term is \( e^{0t} \). The eigenvalues come from the coefficients in the exponentials, 2 and 0. The eigenpairs are \( \begin{pmatrix} 2, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} \), \( \begin{pmatrix} 0, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{pmatrix} \).

Use this page to start your solution. Attach extra pages as needed, then staple.