Applied Differential Equations 2250

Exam date: Thursday, 3 December, 2009

Instructions: This in-class exam is 50 minutes. Up to 60 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Chapter 5)** Complete all.

(1a) [70%] Write the solution of $x''(t) + 9x(t) = 120\sin(t)$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

Answer:

$x(t) = 15\sin(t) - 5\sin(3t)$

(1b) [30%] Determine the practical resonance frequency ω for the electrical equation $13I'' + 2I' + 39I = 100\omega\cos(\omega t)$.

Answer:

$\omega = \sqrt{1/(LC)} = \sqrt{3}$.

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2. **(Chapter 5)** Complete all.

(2a) [75%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 7 with roots $0, 0,\ -1, -1, -1,\ 2i, -2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 3e^x + 4e^{-x} + 5x^3 + 6\sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The atoms of $f(x)$ are e^x, e^{-x}, x^3, $\sin 2x$. Complete the list, adding the related atoms, to obtain 5 groups, each group having exactly one base atom: (1) e^x, (2) e^{-x}, (3) $1, x, x^2, x^3$, (4) $\cos 2x$, (5) $\sin 2x$. The trial solution is a linear combination of 8 atoms, modified by rules to the new list (1) e^x, (2) x^3e^{-x}, (3) x^2, x^3, x^4, x^5, (4) $x\cos 2x$, (5) $x\sin 2x$.

(2b) [25%] Let $f(x) = 4x^3e^x$. Find a constant-coefficient linear homogeneous differential equation of smallest order which has $f(x)$ as a solution.

Answer:

The atom x^3e^x is constructed from roots $1, 1, 1, 1$, listed according to multiplicity. Then the characteristic polynomial must include factor $(r-1)^4$. The smallest order characteristic polynomial must be a constant multiple of $(r-1)^4 = r^4 - 4r^3 + 6r^2 - 4r + 1$. This characteristic equation belongs to the differential equation $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$.

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3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Display the details of Laplace's method to solve the system for $y(t)$. Don't waste time solving for $x(t)$!

$$\begin{aligned} x' &= x + 2y, \\ y' &= 3x, \\ x(0) &= 0, \quad y(0) = 5. \end{aligned}$$

Suggestion: Solve it with scalar methods.
Alternate method: Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ and Cramer's Rule. Notation: $\mathbf{u}$ is the vector solution of $\mathbf{u}' = A\mathbf{u}$ with components $x(t)$, $y(t)$.

Answer:
The Laplace resolvent equation can be written out to find the relations for $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$. Cramer's rule applies to find $\mathcal{L}(y(t)) = \frac{5(s-1)}{(s-3)(s+2)}$. Then partial fractions and backward table methods determine $y(t) = 3e^{-2t} + 2e^{3t}$. The same method applies to determine $\mathcal{L}(x(t)) = \frac{10}{(s-3)(s+2)}$ and then $x(t) = 2e^{3t} - 2e^{-2t}$.

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{3s + 15}{s(s-3)}.$$

Answer:
$\mathcal{L}(f(t)) = \frac{-5}{s} + \frac{8}{s-3} = \mathcal{L}(-5 + 8e^{3t})$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(f(t)) = \frac{2}{s^3} + \frac{2}{(s-3)(s^2 - 6s + 9)}.$$

Answer:
Use the s-differentiation theorem and the first shifting theorem to get $(-t)^2 f(t) = t^2 + t^2 e^{3t}$ or $f(t) = 1 + e^{3t}$.

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4. **(Chapter 10)** Complete all parts.

(4a) [50%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$e^{-t}\sin \pi t$	$2te^{-t/3}$
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{(2s+5)^2}$	$\dfrac{s+1}{s^2+2s+5}$		

Answer:
Left to right: $\frac{1}{4}te^{-5t/2}$, $e^{-t}\cos 2t$, $\frac{\pi}{(s+1)^2+\pi^2}$, $\frac{2}{(s+1/3)^2}$.

(4b) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 4x(t) = 8e^{-2t}, \quad x(0) = x'(0) = 0.$$

Answer:
$x(t) = \sin(2t) - \cos(2t) + e^{-2t}$.

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5. **(Chapter 6)** Complete all parts.

(5a) [20%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -1 & 6 & 1 & 12 \\ -2 & 7 & -3 & 15 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & -5 & 2 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer:
$1, 5, 2 \pm 5i$

(5b) [40%] Given $A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which has eigenvalues $1, -1, -1$, find all eigenvectors.

Answer:
Two frame sequences are required, one for $\lambda = 1$ and one for $\lambda = -1$. Sequence 1 starts with $\begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, the last frame having just one row of zeros. There is one invented symbol t_1 in the last frame algorithm answer. Taking ∂_{t_1} gives one eigenvector, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Sequence 2 starts with $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, with $\mathbf{rref} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. There is one invented symbol t_1 in the last frame algorithm answer. Taking ∂_{t_1} gives one eigenvector, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. This matrix has no Fourier model, it is not diagonalizable.

(5c) [20%] Suppose a 2×2 matrix A has eigenpairs $\left(e^2, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right), \left(e^3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:
Define $P = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$, $D = \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix}$. Then $AP = PD$.

(5d) [20%] Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 2×2 linear differential system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Find the eigenpairs of the matrix C.

Answer:
The missing exponential in the second term is e^{0t}. The eigenvalues come from the coefficients in the exponentials, 2 and 0. The eigenpairs are $\left(2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$.

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