Name Rhiannon Johnson My Class Time 7:30 AM 2250 Midterm 3 [Ver 1. F2009]

Applied Differential Equations 2250

Exam date: Wednesday, 2 December, 2009

Instructions: This in-class exam is 50 minutes. Up to 60 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

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1. (Chapter 5) Complete all.

(1a) [70%] Write the solution of $x''(t) + 16x(t) = 60\sin(t)$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form.**

(1b) [30%] Determine the practical resonance frequency ω for the electrical equation $I'' + 2I' + 5I = 50\omega\cos(\omega t)$.

(a) Trial solution: $x(t) = A\cos t + B\sin t$

$x'(t) = -A\sin t + B\cos t$

$x''(t) = -A\cos t - B\sin t$

$-A\cos t - B\sin t + 16A\cos t + 16B\sin t = 60\sin t$

$15A\cos t + 15B\sin t = 60\sin t$

$15A = 0 \quad A = 0$

$15B = 60 \quad B = 4$

$x(t) = 4\sin t \quad x(0) = 0$

$x'(t) = 4\cos t \quad x'(0) = 4 \implies$ Doesn't satisfy initial conditions

∘ Try with Laplace theory on separate page (see attached page)

ans: $\boxed{x(t) = 4\sin t - \sin 4t}$

(b) For electrical: $LI'' + KI' + \frac{1}{C}I = \omega E_0\cos(\omega t)$?

mechanical: $\omega = \sqrt{k/m - \frac{c^2}{2m^2}}$

$\omega = \frac{1}{\sqrt{LC}} \qquad \omega = \frac{1}{\sqrt{1\cdot 1/5}} = \boxed{\sqrt{5}} = \omega$

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$$\mathcal{L}(x''(t)) + 16\mathcal{L}(x(t)) = 60\mathcal{L}(\sin(t))$$

$$\mathcal{L}(x'(t)) - x'(0)$$

$$s\mathcal{L}(x(t)) - \cancel{x(0)}^{0}) - \cancel{x'(0)}^{0} + 16\mathcal{L}(x(t)) = 60\mathcal{L}(\sin t)$$

$$(s^2+16)\mathcal{L}(x(t)) = 60\left(\frac{1}{s^2+1}\right)$$

$$\mathcal{L}(x(t)) = \frac{60}{(s^2+1)(s^2+16)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+16}$$

$$60 = (As+B)(s^2+16) + (Cs+D)(s^2+1)$$

$$60 = \cancel{As^3} + \cancel{16As} + \cancel{Bs^2} + 16B + \cancel{Cs^3} + \cancel{Cs} + \cancel{Ds^2} + D$$

$$60 = s^3(A+C) + s^2(B+D) + s(C+16A) + 16B + D$$

$$A + C = 0$$

$$B + D = 0 \qquad B = -D$$

$$C + 16A = 0$$

$$16B + D = 60$$

$$15B = 60 \qquad B = 4 \quad D = -4$$

$$\mathcal{L}(x(t)) = \frac{4}{s^2+1} - \frac{4}{s^2+16}$$

$$\mathcal{L}(x(t)) = 4\mathcal{L}(\sin t) - 4\mathcal{L}\left(\frac{\sin 4t}{4}\right)$$

$$\cancel{\mathcal{L}}(x(t)) = 4\cancel{\mathcal{L}}(\sin t) - \cancel{\mathcal{L}}(\sin 4t)$$

Lerch's thm

$$\boxed{x(t) = 4\sin t - \sin 4t}$$

Check: $x(0) = 0 - 0$ ✓

$x'(0) = 4\cos(0) - 4\cos(4\cdot 0)$

$4 - 4 = 0$ ✓

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2. (**Chapter 5**) Complete all.

(2a) [75%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 7 with roots $0, 0, 0, -1, -1, 2i, -2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 2e^x + 3e^{-x} + 4x^3 + 5\cos 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(2b) [25%] The general solution of a certain linear homogeneous differential equation with constant coefficients is

$$y = c_1e^{-2x} + c_2xe^{-2x} + c_3 + c_4x + c_5x^2 + c_6e^x.$$

Find the factored form of the characteristic polynomial.

(a) group 1: e^x

group 2: $\cancel{e^{-x}}, \cancel{xe^{-x}}, x^2e^{-x}$

group 3: $\cancel{1}, \cancel{x}, \cancel{x^2}, x^3, x^4, x^5, x^6$

group 4: $\cancel{\cos 2x}, \cancel{\sin 2x}, x\cos 2x, x\sin 2x$

roots: $0, 0, 0, -1, -1, 2i, -2i$

atoms in homogeneous eq.:

$1, x, x^2, e^{-x}, xe^{-x}, \cos 2x, \sin 2x$

75

Shortest trial solution: $\boxed{y_p = c_1e^x + c_2x^2e^{-x} + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x\cos 2x + c_8x\sin 2x}$

(b) $y = c_1e^{-2x} + c_2xe^{-2x} + c_3 + c_4x + c_5x^2 + c_6e^x$

roots: $-2, -2, 0, 0, 0, 1$

characteristic eq.: $\boxed{r^3(r+2)^2(r-1) = 0}$

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3. (Chapter 10) Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Display the details of Laplace's method to solve the system for $y(t)$. Don't waste time solving for $x(t)$!

Suggestion: Save effort by using the Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ and Cramer's Rule. Notation: $\mathbf{u}$ is the vector solution of $\mathbf{u}' = A\mathbf{u}$ with components $x(t)$, $y(t)$.

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= x, \\ x(0) &= 0, \quad y(0) = 4. \end{aligned}$$

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{4s+20}{s(s+4)}.$$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(f(t)) = \frac{2}{s^3} + \frac{s}{s^2-2s+1}. \qquad \frac{s}{(s-1)^2}$$

a) $\vec{u}' = A\vec{u}$ $\quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $\quad \vec{u}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $\quad A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

$(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0) \implies \begin{pmatrix} s-2 & -3 \\ -1 & s \end{pmatrix}\mathcal{L}(\vec{u}) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

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$\mathcal{L}(\vec{u}) = \begin{pmatrix} s-2 & -3 \\ -1 & s \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} s & 3 \\ 1 & s-2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} \cdot \frac{1}{s(s-2)-3} = \begin{pmatrix} 12 \\ 4s-8 \end{pmatrix}\frac{1}{s^2-2s-3}$

$\mathcal{L}(y(t)) = \frac{4s-8}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$ $\quad A = 1$, $B = 3$ $\quad$ check: $4s-8 = s(A+B) + A - 3B$; $4s-8 = s(4) + 1 - 9$ ✓

$\cancel{\mathcal{L}}(y(t)) = \frac{1}{s-3} + \frac{3}{s+1} = \cancel{\mathcal{L}}(e^{3t}) + 3\cancel{\mathcal{L}}(e^{-t})$

$\boxed{y(t) = e^{3t} + 3e^{-t}}$

b) $\mathcal{L}(f(t)) = \frac{4s+20}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$ $\quad A = 5$, $B = -1$ $\quad$ 25 $\quad \cancel{\mathcal{L}}(f(t)) = \frac{5}{s} - \frac{1}{s+4} = 5\cancel{\mathcal{L}}(1) - \cancel{\mathcal{L}}(e^{-4t})$

$\boxed{f(t) = 5 - e^{-4t}}$

c) $\mathcal{L}(t^2 f(t)) = \mathcal{L}(t^2) + \boxed{\frac{s}{(s-1)^2}} \longrightarrow \frac{s+1}{s^2}\Big|_{s=s-1} = \frac{1}{s} + \frac{1}{s^2} = \mathcal{L}(e^t(1+t))$

$\cancel{\mathcal{L}}(t^2 f(t)) = \cancel{\mathcal{L}}(t^2) + \cancel{\mathcal{L}}(e^t(1+t))$ $\quad \boxed{f(t) = 1 + \frac{e^t(1+t)}{t^2}}$ $\quad$ 25

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98% 100 JBJ

4. (Chapter 10) Complete all parts.

(4a) [50%] Fill in the blank spaces in the Laplace table:

	A	B	C	D	E
$f(t)$	t^3	$\frac{te^{-2/3}}{9}$ ✓	$e^{-t}\cos 3t$ ✓	$e^{-t}\sin \pi t$	$2te^{t/2}$
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{(3s+2)^2}$	$\frac{s+1}{s^2+2s+10}$	$\frac{\pi}{(s+1)^2+\pi^2}$ ✓	$\frac{2}{(s-\frac{1}{2})^2}$

−2 Excused JBJ

(4b) [50%] Solve by Laplace's method for the solution $x(t)$:

(b)

$$\mathcal{L}\,x''(t) + \mathcal{L}\,4x(t) = \mathcal{L}\,10e^{-t}, \quad x(0) = x'(0) = 0.$$

$$s\mathcal{L}(x'(t)) - \cancel{x'(0)}^{0} + 4\mathcal{L}(x(t)) = 10\mathcal{L}(e^{-t})$$

$$s(s\mathcal{L}(x(t)) - \cancel{x(0)})^{0} + 4\mathcal{L}(x(t)) = 10\mathcal{L}(e^{-t})$$

$$(s^2+4)\mathcal{L}(x(t)) = 10\left(\frac{1}{s+1}\right)$$

$$\mathcal{L}(x(t)) = \frac{10}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$10 = \cancel{As^2} + 4A + \cancel{Bs^2} + \cancel{Cs} + \cancel{Bs} + C$$

$$10 = s^2(A+B) + s(C+B) + 4A + C$$

$A + B = 0$　　$-B = C$　　$4A + C = 10$

$5A = 10$　　　　$C = 10 - 4A$

$A = 2 \quad B = -2 \quad C = 2$　　$4A - 10 = B$

$$\mathcal{L}(x(t)) = \frac{2}{s+1} - \frac{2s}{s^2+4} + \frac{2}{s^2+4}$$

$$= 2\mathcal{L}(e^{-t}) - 2\mathcal{L}(\cos 2t) + 2\mathcal{L}\left(\frac{\sin 2t}{2}\right)$$

$$\boxed{x(t) = 2e^{-t} - 2\cos 2t + \sin 2t}$$

✓

(a) B: $\frac{1}{(3s+2)^2} = \frac{1}{9s^2+12s+4} = \frac{1}{9(s+\frac{4}{3}s+\frac{4}{9})} = \frac{1}{9(s+\frac{2}{3})^2}$ $[s+\frac{2}{3} \to s]$ $= \frac{1}{9s^2} = \frac{1}{9}\mathcal{L}(t\cdot e^{-2/3})$

$= \boxed{\frac{te^{-2/3}}{9}}$

C: $\frac{s+1}{s^2+2s+1+9} = \frac{s+1}{(s+1)^2+9}$ $[s+1 \to s] = \frac{s}{s^2+9} = \mathcal{L}(\boxed{\cos 3t \cdot e^{-t}})$

D: $\mathcal{L}(e^{-t}\sin \pi t) = \frac{\pi}{s^2+\pi^2}$ $[s \to s+1] = \boxed{\frac{\pi}{(s+1)^2+\pi^2}}$

E: $2\mathcal{L}(te^{t/2}) = \frac{1}{s}$ $[s \to s-1/2] = \boxed{\frac{1}{s-1/2}}$

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5. (Chapter 6) Complete all parts.

100%

(5a) [20%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -1-\lambda & 6 & 1 & 12 \\ -2 & 7-\lambda & -3 & 15 \\ 0 & 0 & 1-\lambda & 3 \\ 0 & 0 & -3 & 1-\lambda \end{pmatrix}$. To save time, **do not** find eigenvectors!

(5b) [40%] Given $A = \begin{pmatrix} 1-\lambda & 1 & -1 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{pmatrix}$, which has eigenvalues $1, 1, -1$, find all eigenvectors.

(5c) [20%] Suppose a 2×2 matrix A has eigenpairs $\left(\pi, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \left(-\pi, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(5d) [20%] Assume the vector general solution $\vec{u}(t)$ of the 2×2 linear differential system $\vec{u}' = C\vec{u}$ is given by

$$\vec{u}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$\lambda_1 = 2$, $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 0$ $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Find the matrix C.

$$|A - \lambda I| = (-1-\lambda)\begin{vmatrix} 7-\lambda & -3 & 15 \\ 0 & 1-\lambda & 3 \\ 0 & -3 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 6 & 1 & 12 \\ 0 & 1-\lambda & 3 \\ 0 & -3 & 1-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda)(\lambda^2-2\lambda+1+9) + 12(\lambda^2-2\lambda+10) = 0$$

$$(\lambda^2-2\lambda+10)(-7-6\lambda+\lambda^2+12) = 0$$

$$(\lambda^2-2\lambda+10)(\lambda^2-6\lambda+5) = 0$$

$$\left(\lambda + \boxed{\frac{2 \pm \sqrt{4-4(1)(10)}}{2}}\right)(\lambda-5)(\lambda-1)$$

$$\frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$\boxed{\lambda_1 = 1+3i \quad \lambda_2 = 1-3i \quad \lambda_3 = 5 \quad \lambda_4 = 1}$$

For $\lambda = 1$:

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$b - c = 0$ $b = c$

$-b + c = 0$

$b - c = 0$

$$\boxed{\vec{v}_1 = [0 \ 1 \ 1]^T}$$

a is a free variable, so for $\lambda = 1$,

$$\boxed{\vec{v}_2 = [1 \ 0 \ 0]^T}$$

For $\lambda = -1$:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2a + b - c = 0$

$b + c = 0$

$a = 1$ $b = -1$ $c = 1$

$$\boxed{\vec{v}_3 = [1 \ -1 \ 1]^T}$$

$$\boxed{P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \pi & 0 \\ 0 & -\pi \end{bmatrix}}$$

$CP = PD$ $C = PDP^{-1}$ $P = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot -1$

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$$C = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot -1 = \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} \cdot -1 = \boxed{\begin{bmatrix} -2 & -4 \\ 2 & 4 \end{bmatrix} = C}$$