Name ______________________________ **My Class Time** _____ 2250 Midterm 3 [Ver 1, F2009]

Applied Differential Equations 2250

Exam date: Wednesday, 2 December, 2009

Instructions: This in-class exam is 50 minutes. Up to 60 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Chapter 5)** Complete all.

(1a) [70%] Write the solution of $x''(t)+16x(t) = 60\sin(t)$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

Answer:

$x(t) = 4\sin(t) - \sin(4t)$

(1b) [30%] Determine the practical resonance frequency ω for the electrical equation $I'' + 2I' + 5I = 50\omega\cos(\omega t)$.

Answer:

$\omega = \sqrt{1/(LC)} = \sqrt{5}$.

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2. **(Chapter 5)** Complete all.

(2a) [75%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 7 with roots $0, 0, 0, -1, -1, 2i, -2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 2e^x + 3e^{-x} + 4x^3 + 5\cos 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The atoms of $f(x)$ are e^x, e^{-x}, x^3, $\cos 2x$. Complete the list, adding the related atoms, to obtain 5 groups, each group having exactly one base atom: (1) e^x, (2) e^{-x}, (3) $1, x, x^2, x^3$, (4) $\cos 2x$, (5) $\sin 2x$. The trial solution is a linear combination of 8 atoms, modified by rules to the new list (1) e^x, (2) x^2e^{-x}, (3) x^3, x^4, x^5, x^6, (4) $x\cos 2x$, (5) $x \sin 2x$.

(2b) [25%] The general solution of a certain linear homogeneous differential equation with constant coefficients is

$$y = c_1e^{-2x} + c_2xe^{-2x} + c_3 + c_4x + c_5x^2 + c_6e^x.$$

Find the factored form of the characteristic polynomial.

Answer:

The atoms are constructed from roots $-2, -2, 0, 0, 0, 1$, listed according to multiplicity. Then $(r + 2)^2$, r^3 and $(r - 1)$ are factors. The characteristic polynomial is $a(r + 2)^2r^3(r - 1)$ for some nonzero constant a.

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3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Display the details of Laplace's method to solve the system for $y(t)$. Don't waste time solving for $x(t)$!

Suggestion: Save effort by using the Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ and Cramer's Rule. Notation: $\mathbf{u}$ is the vector solution of $\mathbf{u}' = A\mathbf{u}$ with components $x(t)$, $y(t)$.

$$\begin{aligned} &x' = 2x + 3y, \\ &y' = x, \\ &x(0) = 0, \quad y(0) = 4. \end{aligned}$$

Answer:

The Laplace resolvent equation can be written out to find the relations for $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$. Cramer's rule applies to find $\mathcal{L}(y(t)) = \frac{4(s-2)}{(s-3)(s+1)}$. Then partial fractions and backward table methods determine $y(t) = 3e^{-t} + e^{3t}$. The same method applies to determine $\mathcal{L}(x(t)) = \frac{12}{(s-3)(s+1)}$ and then $x(t) = 3e^{3t} - 3e^{-t}$.

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{4s + 20}{s(s+4)}.$$

Answer:

$\mathcal{L}(f(t)) = \frac{5}{s} + \frac{-1}{s+4} = \mathcal{L}(5 - e^{-4t})$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(f(t)) = \frac{2}{s^3} + \frac{s}{s^2 - 2s + 1}.$$

Answer:

Use the s-differentiation theorem and the first shifting theorem to get $(-t)^2 f(t) = t^2 + (t+1)e^t$ or $f(t) = 1 + \frac{(t+1)e^t}{t^2}$.

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4. **(Chapter 10)** Complete all parts.

(4a) [50%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$e^{-t}\sin \pi t$	$2te^{t/2}$
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{(3s+2)^2}$	$\dfrac{s+1}{s^2+2s+10}$		

Answer:
Left to right: $\frac{1}{9}te^{-2t/3}$, $e^{-t}\cos 3t$, $\frac{\pi}{(s+1)^2+\pi^2}$, $\frac{2}{(s-1/2)^2}$.

(4b) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 4x(t) = 10e^{-t}, \quad x(0) = x'(0) = 0.$$

Answer:
$x(t) = \sin(2t) - 2\cos(2t) + 2e^{-t}$.

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5. (Chapter 6) Complete all parts.

(5a) [20%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -1 & 6 & 1 & 12 \\ -2 & 7 & -3 & 15 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 1 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer:
$1, 5, 1 \pm 3i$

(5b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which has eigenvalues $1, 1, -1$, find all eigenvectors.

Answer:
Two frame sequences are required, one for $\lambda = 1$ and one for $\lambda = -1$. Sequence 1 starts with $\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, the last frame having two rows of zeros. There are two invented symbols t_1, t_2 in the last frame algorithm answer. Taking ∂_{t_1} and ∂_{t_2} gives two eigenvectors, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Sequence 2 starts with $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, with $\mathbf{rref} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. There is one invented symbol t_1 in the last frame algorithm answer. Taking ∂_{t_1} gives one eigenvector, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(5c) [20%] Suppose a 2×2 matrix A has eigenpairs $\left(\pi, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, $\left(-\pi, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:
Define $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}$. Then $AP = PD$.

(5d) [20%] Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 2×2 linear differential system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Find the matrix C.

Answer:
The missing exponential in the second term is e^{0t}. The eigenvalues come from the coefficients in the exponentials, 2 and 0. The eigenpairs are $\left(2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$, $\left(0, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Solve $CP = PD$ to find $C = \begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}$.

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