Name KEY

2250 Midterm 2 [Ver 2, F2009]

Applied Differential Equations 2250

Exam date: Thursday, 5 November, 2009

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 possibilities)**
Determine A, B such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & -2B \\ 5x & + & By & + & 4z & = & A-B \\ -2x & - & 4y & - & 3z & = & 2-B \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2B \\ 5 & B & 4 & A-B \\ -2 & -4 & -3 & 2-B \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2B \\ 0 & B-10 & -6 & A+9B \\ 0 & 0 & 1 & 2-5B \end{array}\right)$$ combo(1,2,−5) combo(1,3,2)

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2B \\ 0 & B-10 & 0 & 12+A-21B \\ 0 & 0 & 1 & 2-5B \end{array}\right)$$ combo(3,2,6)

Unique solution: $B \neq 10$

Sanity test: should be $\det \neq 0$

No solution: $B = 10$ $\quad 12 + A - 21B \neq 0$ [or $A \neq 198$]

∞-many solutions: $B = 10$, $A = 198$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY 2250 Midterm 2 [Ver 2, F2009]

2. (vector spaces) Do all parts.

(a) [20%] State one theorem which has conclusion *S is a subspace.*

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the restriction equations $\ln(1+x_1^2)=0$, $(4x_2-x_3)^2=0$. Prove or disprove that S is a subspace of V.

(c) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & - & x_3 & + & 2x_4 & = & 0, \\ x_1 & + & 2x_2 & - & 2x_3 & + & 2x_4 & = & 0, \\ & & 2x_2 & + & 2x_3 & & & = & 0, \\ x_1 & + & 4x_2 & & & + & 2x_4 & = & 0. \end{array}$$

(a) Kernel Theorem, If The restriction equations defining S can be written as a matrix equation $A\vec{x}=\vec{0}$, Then S is a subspace.

Subspace criterion, A subset S of a vector space V is a subspace if (1) $\vec{0}$ in S, (2) $\vec{x},\vec{y}$ in $S \Rightarrow \vec{x}+\vec{y}$ in S, (3) $c=$ scalar and $\vec{x}$ in $S \Rightarrow c\vec{x}$ in S.

(b) Because $x^2=0 \Leftrightarrow x=0$, Then $\ln(1+x_1^2)=0 \Leftrightarrow x_1=0$ and $(4x_2-x_3)^2=0 \Leftrightarrow 4x_2-x_3=0$. Define $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, Then $A\vec{x}=\vec{0}$ is equivalent to The restriction equations. Set S is a subspace by The Kernel Theorem.

(c) The rref is $\begin{pmatrix} 1 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which implies $\begin{cases} x_1 = 4t_1 - 2t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$

The basis is $\left\{ \frac{\partial \vec{x}}{\partial t_1}, \frac{\partial \vec{x}}{\partial t_2} \right\} = \left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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Name KEY

3. (independence) Do all parts.

(a) [25%] Let $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, State a test that can decide independence or dependence of this list of three vectors [10%]. Apply the test and report the result [15%].

(b) [25%] Let $f_1(x) = x^2$, $f_2(x) = x^{1/2}$. State a test which can decide independence or dependence of the two functions on $0 < x < \infty$ [10%]. Apply the test and report the result [15%].

(c) [50%] Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 7 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 0 \\ 5 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}.$$

(a) Test. A = augmented matrix of u_1, u_2, u_3. Then $\{u_1, u_2, u_3\}$ is independent $\iff$ rank$(A) = 3$

Apply. $A = \begin{pmatrix} 3 & 5 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ has at most rank 2 $\Rightarrow$ dependent.

(b) Wronskian Test: $\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} \neq 0$ for some $x \Rightarrow \{f_1, f_2\}$ indep.

Sample test: $\begin{vmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{vmatrix} \neq 0$ for some $x_1, x_2 \Rightarrow$ ''

Apply: $\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{1/2} \\ 2x & \frac{1}{2}x^{-1/2} \end{vmatrix} = -\frac{3}{2}x^{3/2} \neq 0$ for $x=1 \Rightarrow \{f_1, f_2\}$ indep.

(c) $\text{rref} = \begin{pmatrix} 1 & 0 & 8/3 & 0 & 3 \\ 0 & 1 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ by pivot Theorem $\vec{a}, \vec{b}, \vec{d}$ form a largest set of indep. vectors.

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Name KEY

4. **(determinants)** Do all parts.

(a) [40%] Assume given 3×3 matrices A, B. Suppose $E_5E_4BA = E_3E_2E_1$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a multiply by -5, a combination, a multiply by 3, and a swap. Assume $\det(A) = 4$. Find $\det(2AB)$.

(b) [20%] Define $A = \begin{pmatrix} -1 & 6x-1 & 9x-5 \\ 2 & 0 & 2 \\ 15x & 0 & -1 \end{pmatrix}$. Determine all values of x for which $(I+B)^{-1}$ fails to exist, where B is the transpose of A and I is the identity matrix.

(c) [40%] Let matrix A be defined as below. Apply the adjugate formula for the inverse to find a determinant formula for the entry of A^{-1} located in row 2, column 3. Other methods are not acceptable. To save time, **do not evaluate determinants**.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) The determinant product Theorem and known values of determinants of elementary matrices imply

$$(-1)(3)|B|(4) = (1)(-5)(-1)$$
$$|B| = -\frac{5}{12}$$

$$\det(2AB) = \det(2I)\det(A)\det(B) = 2^3(4)(-5/12) = \boxed{\frac{-40}{3}}$$

(b) $|I+A| = \begin{vmatrix} 0 & 6x-1 & 9x-5 \\ 2 & 1 & 2 \\ 15x & 0 & 0 \end{vmatrix} = 15x(3x+3)$ $\boxed{\begin{matrix} x=0 \\ x=-1 \end{matrix}}$

used Thm: $|I+A| = |(I+A)^T| = |I+B|$

(c) $\text{entry} = \dfrac{\text{cofactor}(A,3,2)}{|A|} = \dfrac{(-1)}{|A|}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \dfrac{-2}{|A|} = \boxed{\frac{1}{3}}$

$|A| = 1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 2 - 8 = -6$

cofactor exp on column 4

cofactor exp on row 1

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Name KEY

5. **(Linear differential equations)** Do all three parts.

(a) [30%] Find a solution basis for $6y'' + 17y' + 5y = 0$ and report the general solution.

(b) [40%] The characteristic equation is $r^2(r^2 + 5r + 6)(r^2 + 2r + 5)^2 = 0$. Find the general solution y of the homogeneous constant-coefficient differential equation.

(c) [30%] A second order differential equation $y'' + py' + qy = 0$ with constant coefficients p, q has general solution $y = c_1e^{-x/2} + c_2(e^x + 2e^{-x/2})$. Find p and q.

(a) Roots $= -\frac{5}{2}, -\frac{1}{3}$; atoms $= e^{-5x/2}, e^{-x/3}$ = Solution basis

General solution = linear combination of the atoms

(b) $r = 0, 0, -3, -2, -1 \pm 2i, -1 \pm 2i$

atoms $= 1, x, e^{-3x}, e^{-2x}, e^{-x}\cos 2x, xe^{-x}\cos 2x, e^{-x}\sin 2x, xe^{-x}\sin 2x$

y = linear combination of these 8 atoms

(c) The DE must have atoms $e^x, e^{-x/2}$, hence $r^2+pr+q=0$ has roots $1, -1/2$.

$(r-1)(r+\frac{1}{2}) = r^2+pr+q$

$r^2 - \frac{1}{2}r - \frac{1}{2} = r^2+pr+q$

$\boxed{p = q = -\frac{1}{2}}$

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Name KEY

Applied Differential Equations 2250

Exam date: Thursday, 5 November, 2009

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 possibilities)**
Determine A, B such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & -2B \\ 5x & + & Ay & + & 4z & = & -B \\ -2x & - & 4y & - & 3z & = & 2-B \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2B \\ 5 & A & 4 & -B \\ -2 & -4 & -3 & 2-B \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2B \\ 0 & A-10 & -6 & 9B \\ 0 & 0 & 1 & 2-5B \end{array}\right) \begin{array}{l} \text{combo}(1,2,-5) \\ \text{combo}(1,3,2) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2B \\ 0 & A-10 & 0 & 12-21B \\ 0 & 0 & 1 & 2-5B \end{array}\right) \text{combo}(3,2,6)$$

Unique solution: $A \neq 10$ same as $\det \neq 0$

No solution: $A = 10$, $B \neq 4/7$

∞-many solutions: $A = 10$, $B = 4/7$

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Name KEY 2250 Midterm 2 [Ver 3, 7:30, F2009]

2. (vector spaces) Do all parts.

(a) [20%] Define what it means for S to be a subspace of V.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the restriction equations $e^{x_1} = 1$, $x_2 = 2x_3$. Prove or disprove that S is a subspace of V.

(c) [40%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & - & x_3 & - & 3x_4 & = & 0, \\ x_1 & + & 2x_2 & - & 2x_3 & - & 3x_4 & = & 0, \\ & & 2x_2 & + & 2x_3 & & & = & 0, \\ x_1 & + & 4x_2 & & & - & 3x_4 & = & 0. \end{array}$$

(a) S is a subspace of vector space V if it is a nonvoid subset of V that is a vector space under the operations $+, \cdot$ of V.

(b) $e^{x_1} = 1$ and $x_2 = 2x_3 \iff \begin{cases} x_1 = 0 \\ x_2 - 2x_3 = 0 \end{cases}$

$\iff \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff A\vec{x} = \vec{0}$ for matrix A above.

By the Kernel Theorem, S is a subspace.

(c) The rref $= \begin{pmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 4t_1 + 3t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$

Basis $= \left\{ \frac{\partial \vec{x}}{\partial t_1}, \frac{\partial \vec{x}}{\partial t_2} \right\} = \left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

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Name KEY

3. **(independence)** Do all parts.

(a) [25%] Let $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, State a test that can decide independence or dependence of this list of three vectors [10%]. Apply the test and report the result [15%].

(b) [25%] Let $f_1(x) = x$, $f_2(x) = x^{3/2}$. State a test which can decide independence or dependence of the two functions on $0 < x < \infty$ [10%]. Apply the test and report the result [15%].

(c) [50%] Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 7 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 0 \\ 15 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}.$$

(a) Test: Let A = augmented matrix of u_1, u_2, u_3. Then $\{u_1, u_2, u_3\}$ is independent $\Leftrightarrow$ rank$(A) = 3$

Apply: $A = \begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$ has rank $\le 2 \Rightarrow$ dependent

(b) Wronskian Test: $\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} \ne 0$ for some $x \Rightarrow \{f_1, f_2\}$ indep.

Apply: $\begin{vmatrix} x & x^{3/2} \\ 1 & \frac{3}{2}x^{1/2} \end{vmatrix} = \frac{1}{2}x^{3/2} \ne 0$ at $x=1 \Rightarrow \{x, x^{3/2}\}$ indep on $0 < x < \infty$.

(c) RREF $= \begin{pmatrix} 1 & 0 & 8/3 & 0 & 3 \\ 0 & 1 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ largest set of indep columns is $\vec{a}, \vec{b}, \vec{d}$ by the pivot Theorem

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Name KEY

4. **(determinants)** Do all parts.

(a) [40%] Assume given 3×3 matrices A, B. Suppose $E_5 E_4 BA = E_3 E_2 E_1 B$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a multiply by -2, a combination, a multiply by 3, and a swap. Assume $\det(A) = 4$. Find $\det(2AB)$.

(b) [20%] Define $A = \begin{pmatrix} -1 & 2x-1 & 3x-5 \\ 2 & 0 & 2 \\ 5x & 0 & -1 \end{pmatrix}$. Determine all values of x for which $(I+B)^{-1}$ fails to exist, where B is the transpose of A and I is the identity matrix.

(c) [40%] Let matrix A be defined as below. Apply the adjugate formula for the inverse to find the value of the entry of A^{-1} located in row 2, column 3. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

(a) The determinant product theorem plus known determinant values for elementary matrices imply

$$(-1)(3)\,|B||A| = (1)(-2)(-1)\,|B|$$
$$(-3)\,|B|\,(4) = 2\,|B|$$
$$|B| = 0$$

First ans. $\det(2AB) = \det(2I)\det(A)\det(B) = \boxed{0}$

→ At Exam time, The problem was changed, replacing BA by A. Then The answer is

$$(-1)(3)(4) = (1)(-2)(-1)\,|B|$$
$$-6 = |B|$$

Second ans. $\det(2AB) = \det(2I)\det(A)\det(B) = \boxed{2^3(4)(-6)}$

(b) $|I+B| = \begin{vmatrix} 0 & 2x-1 & 3x-5 \\ 2 & 1 & 2 \\ 5x & 0 & 0 \end{vmatrix} = 5x(x+3)$ $\boxed{x=0,\ x=-3}$

(c) $\text{entry} = \dfrac{\text{cofactor}(A,3,2)}{|A|} = \dfrac{(-1)\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}}{6} = \boxed{\dfrac{-1}{3}}$

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Name KEY

5. **(Linear differential equations)** Do all three parts.

(a) [30%] Find a solution basis for $2y'' + 4y' + 10y = 0$ and report the general solution.

(b) [40%] The characteristic equation is $r^3(r^2 - r - 6)(r^2 + 2r + 10)^2 = 0$. Find the general solution y of the homogeneous constant-coefficient differential equation.

(c) [30%] A second order differential equation $y'' + py' + qy = 0$ with constant coefficients p, q has general solution $y = c_1 e^x + c_2(e^x + e^{-x/2})$. Find p and q.

(a) Roots $= -1 \pm 2i$, atoms $= e^{-x}\cos 2x, e^{-x}\sin 2x$ = sol. basis

y = linear combination of the atoms

(b) roots $= 0, 0, 0, 3, -2, -1 \pm 3i, -1 \pm 3i$ (9 roots)

atoms $= 1, x, x^2, e^{3x}, e^{-2x}, e^{-x}\cos 3x, xe^{-x}\cos 3x, e^{-x}\sin 3x, xe^{-x}\sin 3x$

y = linear combination of the 9 atoms

(c) atoms must be $e^x, e^{-x/2}$

roots $= 1, -1/2$

Factors $= r-1, r+1/2$

$(r-1)(r+1/2) = r^2 + pr + q$

$r^2 - \frac{1}{2}r - \frac{1}{2} = r^2 + pr + q$

$\boxed{p = q = -\frac{1}{2}}$

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