Name KEY Ver 1

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [Version 1]

Wednesday, 30 September 2009

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [25%] Solve $y' = \dfrac{1+x^2}{1+2x+x^2}$.

(b) [25%] Solve $y' = (\sin x + \tan x)(\sec^2 x + \cos x)$.

(c) [25%] Solve $y' = x \sec\left(x^2\right) \tan\left(x^2\right)$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^t v(t)) = 0$, $v(0) = 10$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 100$.

(a) $\dfrac{1+x^2}{1+2x+x^2} = \dfrac{1+x^2}{(1+x)^2} = \dfrac{1+(u-1)^2}{u^2} = \dfrac{2+u^2-2u}{u^2} = 2u^{-2}+1-2u^{-1}$

where $u = x+1$. Then $y = \int \frac{1+x^2}{1+2x+x^2}\,dx = -2u^{-1}+u-2\ln|u|+c$

$y = \dfrac{-2}{x+1} + x - 2\ln|x+1| + c_1$ where $c_1 = c+1$.

(b) $y' = uu'$ where $u = \sin x + \tan x \implies y = \dfrac{u^2}{2} + c \implies$

$y = \frac{1}{2}(\sin x + \tan x)^2 + c$

(c) $y' = \frac{1}{2}(\sec u)(\tan u)\, u'$ where $u = x^2 \implies y = \frac{1}{2}\sec u + c$

$\implies y = \frac{1}{2}\sec(x^2) + c$

(d) $e^t v(t) = c$, $e^0 v(0) = c \implies v(t) = 10e^{-t}$

$\dfrac{dx}{dt} = 10e^{-t} \implies x(t) = -10e^{-t} + c_1$, $100 = x(0) = -10e^0 + c_1$

$\implies c_1 = 110$, $x(t) = 110 - 10e^{-t}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY Ver 1

2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form. No details expected.

$\boxtimes$	$y' + xy^2 = y(xy + e^x) + x^2y$	$\boxtimes$	$y' = (x-1)(y+1) - xy - x$
$\boxtimes$	$y' = e^x\left(2e^{x-y}e^{3y} + 3e^{2x+2y}\right)$	$\square$	$y' + e^y = xe^{2y}$

(b) [10%] Is $y' + xy = ye^x$ linear? No details expected.

(c) [10%] Give an example of $y' = f(x,y)$ which is linear but not separable. No details expected.

(d) [40%] Apply tests to show that $y' = x + e^y$ is not separable and not linear. Supply all details.

(a) $y' + xy^2 = xy^2 + ye^x + x^2y \Rightarrow y' = ye^x + x^2y \Rightarrow y' = y(e^x + x^2)$

$y' = 2e^{2x}e^{2y} + 3e^{3x}e^{2y} \Rightarrow y' = e^{2y}(2e^{2x} + 3e^{3x})$

$y' = xy - y + x - 1 - xy - x \Rightarrow y' = -y - 1$

$y' = xe^{2y} - e^y$ not separable by $\dfrac{f_x}{f}$ test

(b) yes, $y' + (x - e^x)y = 0$ linear homogeneous

(c) $y' + y = x$

(d) Not Linear: $\dfrac{\partial f}{\partial y} = e^y$ not indep of $y \Rightarrow$ not linear

Not separable: $\dfrac{f_x}{f} = \dfrac{1}{x+e^y}$ depends on $y \Rightarrow$ not separable

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY Ver 1

3. (Solve a Separable Equation)
Given $(x+4)(y-1)y' = \left((x+4)e^{-x+2} + 2x^2\right)(y+1)^2$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{y-1}{(y+1)^2}y' = e^{-x+2} + \frac{2x^2}{x+4}$$

$$\int \frac{y-1}{(y+1)^2}y'\,dx = \int \left(e^{-x+2} + \frac{2x^2}{x+4}\right)dx$$

$$\begin{cases} u = y+1 \\ du = y'\,dx \end{cases} \qquad \begin{cases} v = x+4 \\ dv = dx \end{cases}$$

$$\int \frac{u-2}{u^2}\,du = -e^{-x+2} + \int \frac{2(v-4)^2}{v}\,dv$$

$$\int \left(\frac{1}{u} - 2u^{-2}\right)du = -e^{-x+2} + \int \left(\frac{2v^2}{v} - \frac{16v}{v} + \frac{32}{v}\right)dv$$

$$\ln|u| + \frac{2}{u} = -e^{-x+2} + \frac{2v^2}{2} - 16v + 32\ln|v| + c_1$$

$$\ln|y+1| + \frac{2}{y+1} = -e^{-x+2} + x^2 + 8x + 16 - 16x - 64 + 32\ln|x+4| + c_1$$

$$\ln|y+1| + \frac{2}{y+1} = -e^{-x+2} + x^2 - 8x + 32\ln|x+4| + c, \qquad c = c_1 - 48$$

✓ ans check: $(\text{LHS})' = \frac{1}{y+1} - \frac{2}{(y+1)^2} = \frac{y-1}{(y+1)^2}$

✓ ans check: $(\text{RHS})' = e^{-x+2} + 2x - 8 + \frac{32}{x+4}$

$$= -e^{-x+2} + \frac{2(x-4)(x+4)+32}{x+4}$$

$$= -e^{-x+2} + \frac{2x^2}{x+4}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY Ver 1

4. (Linear Equations)

(a) [60%] Solve the linear model $5x'(t) = -160 + \frac{25}{t+3}x(t)$, $x(0) = 24$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} - (2x+1)y = 0$.

(c) [20%] Solve $2\frac{dy}{dx} + 5y = 12$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' - \frac{5}{t+3}x = \frac{-160}{5}$

$\frac{(xW)'}{W} = -32 \qquad W = e^{\int \frac{-5}{t+3}dt} = e^{-5\ln|t+3|}$

$W = (t+3)^{-5}$

$(xW)' = -32W$

$xW = -32\int (t+3)^{-5}dt$

$xW = -32\frac{(t+3)^{-4}}{-4} + c$

$x = -32\frac{(t+3)^{-4}}{-4}(t+3)^5 + c(t+3)^5$

$24 = x(0) = 8(3) + c(3)^5$

$\Rightarrow c = 0$

$x(t) = 8(t+3)$

(b) $y = \frac{c}{\text{integ factor}} = \frac{c}{e^{-x^2-x}}$

(c) $y_p = \frac{12}{5} \qquad y_h = \frac{c}{e^{5x/2}} \qquad y = \frac{12}{5} + ce^{-5x/2}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY Ver 1

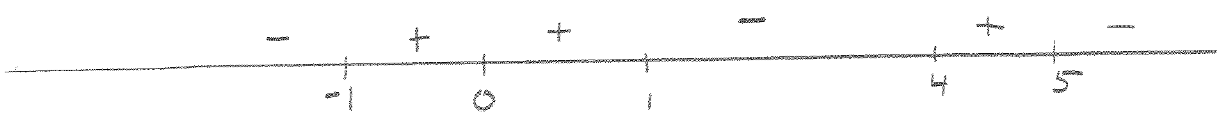
5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = e^x \ln(1 + x^4)\,(3 - |x - 2|)^3\,(1 + x)(4 - x)(1 - x^2)^3.$$

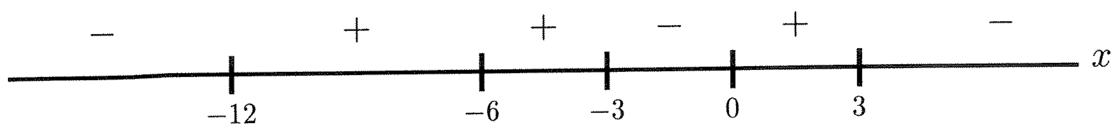
Expected in the phase line diagram are equilibrium points and signs of dx/dt.

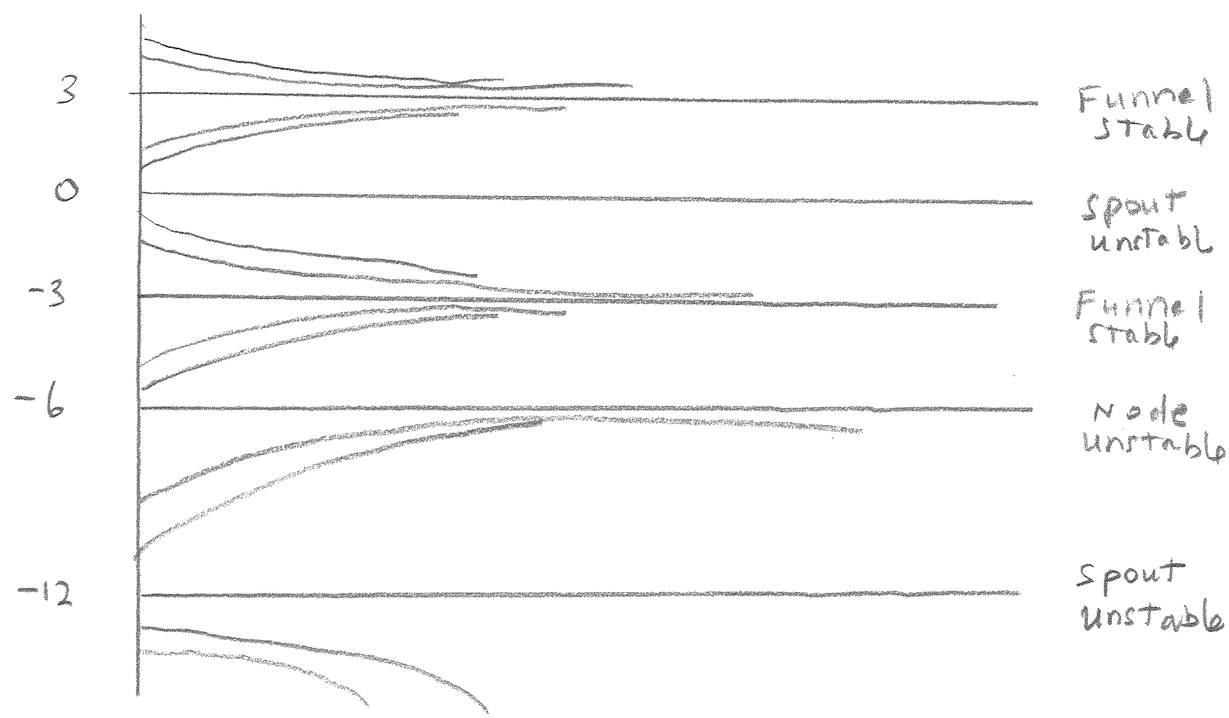
---

equil = 0, 5, -1, 4, 1



(b) [50%] Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.





Use this page to start your solution. Attach extra pages as needed, then staple.