Name ____________________ Class Time _____

Math 2250 Extra Credit Problems
Chapter 8
Fall 2009

Due date: See the internet due date for 9.4, which is the due date for these problems. Records are locked on that date and only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., Xc1.2-4. You may attach this printed sheet to simplify your work.

Problem Xc8.1-4. (Fundamental Matrix)

Find a fundamental matrix $\Phi(t)$ by each of the following methods. Report $e^{At} = \Phi(t)\Phi(0)^{-1}$, using one of the answers for Φ.

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

(a) *Cayley-Hamilton method.* Compute the characteristic equation $\det(A - \lambda I) = 0$. Find two atoms from the roots of this equation. Then $x(t)$ is a linear combination of these atoms. The first equation $x' = 2x - 5y$ can be solved for y to find the second answer.

(b) *Eigenanalysis method.* Find the eigenpairs $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2)$ of A. Let Φ have columns $e^{\lambda_1 t}\mathbf{v}_1$, $e^{\lambda_2 t}\mathbf{v}_2$.

Problem Xc8.1-12. (Putzer's Method)

The exponential matrix e^{At} can be found in the 2×2 case from Putzer's formula

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ are equal, then compute the Newton quotient factor by L'Hopital's rule, limiting $\lambda_2 \to \lambda_1$ [λ_1, t fixed]. If the roots are complex, then take the real part of the right side of the equation.

Compute from Putzer's formula e^{At} for the following cases.

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Answer $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc8.1-38. (Laplace's Method)

The exponential matrix e^{At} can be found from the Laplace resolvent formula for the problem $\Phi' = A\Phi$, $\Phi(0) = I$:

$$\mathcal{L}(\Phi(t)) = (sI - A)^{-1}\Phi(0) = (sI - A)^{-1}.$$

For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives $\mathcal{L}(e^{At}) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix}$, which implies $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

Compute $\Phi(t) = e^{At}$ using the resolvent formula for the following cases.

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc8.2-4. **(Variation of Parameters)**

Use the variation of parameters formula $\mathbf{u}_p(t) = e^{At} \int e^{-At}\mathbf{f}(t)dt$ to find a particular solution of the given system. Please use `maple` to do the indicated integration, following the example below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$.

Example: Solve for $\mathbf{u}_p(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t),expAt(-t).f(t));
up:=simplify(expAt(t).integral);
```

Problem Xc8.2-19. **(Initial Value Problem)**

Solve the given initial value problem using a computer algebra system. Follow the example given below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example: Solve for $\mathbf{u}(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The answer is $\mathbf{u} = \begin{pmatrix} -e^{-t} \\ e^{-t} - 1 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t=0..t),expAt(-t).f(t));
up:=unapply(expAt(t).integral,t):
u0:=Vector([-1,0]);
uh:=t->expAt(t).(u0-up(0));
u:=simplify(uh(t)+up(t));
```

End of extra credit problems chapter 8.